CLASSIFICATION OF p-ADIC FUNCTIONS SATISFYING KUMMER TYPE CONGRUENCES

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Abstract. We introduce p-adic Kummer spaces of continuous functions on $\mathbb{Z}_p$ that satisfy certain Kummer type congruences. We will classify these spaces and show their properties, for instance, ring properties and certain decompositions. As a result, these functions have always a fixed point, functions of certain subclasses have always a unique simple zero in $\mathbb{Z}_p$. The fixed points and the zeros are effectively computable by given algorithms. This theory can be transferred to values of Dirichlet L-functions at negative integer arguments. That leads to a conjecture about their structure supported by several computations. In particular we give an application to the classical Bernoulli and Euler numbers. Finally, we present a link to p-adic functions that are related to Fermat quotients.

Contents

1. Introduction

Throughout this paper p denotes a prime. The author [17] showed some special results for p-adic zeta functions, introduced by Kubota and Leopoldt [20], see especially Koblitz [19]. These functions interpolate values of divided Bernoulli numbers in certain residue classes, which are values of the Riemann zeta function at negative integers, modified by an Euler factor.

2000 *Mathematics Subject Classification.* Primary 11A07; Secondary 11A25, 11B65, 11B68.

Key words and phrases. p-adic functions, p-adic Kummer spaces, Kummer congruences, finite differences, Bernoulli numbers, Euler numbers, Fermat quotients.

To avoid confusion, the p-adic L-function $L_p(s,\chi)$ in context of Iwasawa theory is the second construction of Kubota-Leopoldt, while we only consider their first construction.

Although there is a vast literature about Kummer congruences and its generalizations for Bernoulli numbers and other special sequences, commonly called Kummer type congruences, the results are presented in their contexts.

We will here establish a generalized theory, using new methods, for arbitrary p-adic functions that satisfy certain Kummer type congruences. This is embedded in the theory of continuous functions on $\mathbb{Z}_p$, which always have a Mahler expansion. Therefore we introduce the p-adic Kummer spaces $\mathcal{K}_{p,1}$ and $\mathcal{K}_{p,2}$ of such functions and show their relations and properties, for instance, ring properties, certain decompositions, and that $\mathcal{K}_{p,2} \subsetneq \mathcal{K}_{p,1}$.

As a result, functions of $\mathcal{K}_{p,1}$ and $\mathcal{K}_{p,2}$ have always a fixed point in $\mathbb{Z}_p$. Functions of an important subclass $\widehat{\mathcal{K}}^0_{p,2} \subset \mathcal{K}_{p,2}$ have always a unique simple zero in $\mathbb{Z}_p$. A product of the latter functions provides a product of linear terms when viewed in the p-adic norm. We present two algorithms, which can effectively compute a truncated p-adic expansion of the fixed point of a function of $\mathcal{K}_{p,2}$, resp., of the zero of a function of $\widehat{\mathcal{K}}^0_{p,2}$.

All results of the p-adic Kummer spaces can be transferred back to values of ordinary L-functions at negative integer arguments, which are associated with a real Dirichlet character. Since these functions, modified by an Euler factor, obey the Kummer type congruences in certain residue classes, we obtain p-adic L-functions of $\mathcal{K}_{p,2}$. In contrast, the construction of Kubota-Leopoldt yields a p-adic L-function of $\mathcal{K}_{p,1}$ in our terminology. As a special case, we apply these results to the classical Bernoulli and Euler numbers.

At the end, we construct p-adic functions using Fermat quotients, which have a similar behavior as the p-adic zeta and L-functions mentioned above.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ be the set of positive integers, the set of primes, the ring of integers, the field of rational, real, and complex numbers, respectively. Let $\mathbb{Z}_p$ be the ring of p-adic integers and $\mathbb{Q}_p$ be the field of p-adic numbers. The ultrametric absolute value $|\cdot|_p$ is defined by $|s|_p = p^{-\operatorname{ord}_p s}$ on $\mathbb{Q}_p$. Let $|\cdot|_\infty = |\cdot|$ be the usual norm on $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{C}$. Following [26, Ch. 4–5], we denote $\mathcal{C}(\mathbb{Z}_p)$, $\operatorname{Lip}(\mathbb{Z}_p)$, and $\mathcal{S}^1(\mathbb{Z}_p)$ as the space of continuous, Lipschitz, and strictly differentiable functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$, respectively. For continuous functions $f \in \mathcal{C}(\mathbb{Z}_p)$ define the norm $\|f\|_p = \sup_{s \in \mathbb{Z}_p} |f(s)|_p = \max_{s \in \mathbb{Z}_p} |f(s)|_p$ on the compact space $\mathbb{Z}_p$. Let $\mathcal{O}$ denote the Landau symbol.

Definition 2.1. *The linear forward difference operator ∇_h with increment h and its powers are defined by*

$$\nabla_h^n f(s) = \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{n-\nu} f(s+\nu h)$$

for integers $n \geq 0$, $h \geq 1$, and any function $f : \mathbb{Z}_p \to \mathbb{Z}_p$. For brevity we write ∇^n instead of ∇_1^n. In case of ambiguity we explicitly indicate the variable with a possible start value,

for example $t = 0$, by the expression

$$\nabla_h^n f(s+t)\,\Big|_{t=0}.$$

The falling factorials are defined by

$$(s)_0 = 1, \quad (s)_n = s(s-1)\cdots(s-n+1) \quad \text{for } n \geq 1.$$

As usual, let $\binom{s}{n} = (s)_n/n!$ for $n \geq 0$ be the binomial polynomial, which is a function on $\mathbb{Z}_p$. A series

$$f(s) = \sum_{\nu \geq 0} a_\nu \binom{s}{\nu}$$

with coefficients $a_\nu \in \mathbb{Z}_p$, where $|a_\nu|_p \to 0$, is called a Mahler series, which defines a continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$.

Theorem 2.2 (Mahler [24])**.** *If $f \in \mathcal{C}(\mathbb{Z}_p)$, then f has a unique Mahler expansion*

$$f(s) = \sum_{\nu \geq 0} a_\nu \binom{s}{\nu},$$

where the coefficients $a_\nu \in \mathbb{Z}_p$ are given by $a_\nu = \nabla^\nu f(0)$ with $|a_\nu|_p \to 0$.

Lemma 2.3 ([26, Ch. 5.1, p. 227])**.** *Let $k \geq 1$ and $p^j \leq k < p^{j+1}$. We have*

$$\left| \binom{s}{k} - \binom{t}{k} \right|_p \leq p^j \, |s-t|_p, \quad s, t \in \mathbb{Z}_p.$$

Theorem 2.4. *Let h, m, n be positive integers with $m \geq n$. Then*

$$\operatorname{ord}_p \left(p^m \, \nabla_h^n \binom{s}{m} \right) \geq n(1 + \operatorname{ord}_p h), \quad s \in \mathbb{Z}_p.$$

We will prove this theorem in the end of this section, since we shall need several preparations. For basic properties of differences see, for instance, [13] and [26].

Lemma 2.5 ([26, Ch. 3.1, p. 241])**.** *Let n be a positive integer. We have*

$$\operatorname{ord}_p n! = \frac{n - S_p(n)}{p-1} \geq 0 \quad \text{and} \quad \operatorname{ord}_p \left(\frac{p^n}{n!} \right) = \frac{p-2}{p-1} n + \frac{S_p(n)}{p-1} \geq 1,$$

where $S_p(n)$ is the sum of the digits of the p-adic expansion of n.

Lemma 2.6. *Let $f \in \mathbb{Q}_p[s]$ a function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ with $m = \deg f$ and $a_m \in \mathbb{Q}_p$ be the highest coefficient of f. For positive integers h, n and $s \in \mathbb{Z}_p$ we have*

$$\nabla_h^n f(s) = \begin{cases} h^n \, g(s), & n < m, \\ h^m \, m! \, a_m, & n = m, \\ 0, & n > m, \end{cases}$$

where $m! \, a_m, h^n \, g(s) \in \mathbb{Z}_p$ and in the latter case $g \in \mathbb{Q}_p[s]$ with $\deg g = m - n$. If $f \in \mathbb{Z}_p[s]$, then $g \in \mathbb{Z}_p[s]$.

Proof. We have $\nabla_h s = h$ and $\nabla_h s^m = h(ms^{m-1} + \mathcal{O}(s^{m-2}))$ for $m \geq 2$, while constant terms vanish under ∇_h. Case $n < m$: We get $\nabla_h f(s) = h\, \tilde{g}(s)$ with $\deg \tilde{g} = m - 1$ and by iteration that $\nabla_h^n f(s) = h^n g(s)$ with $\deg g = m - n$. Since f takes only values in $\mathbb{Z}_p$, so also its differences. Thus $h^n g(s) \in \mathbb{Z}_p$. If $f \in \mathbb{Z}_p[s]$, then ∇_h maps coefficients from $\mathbb{Z}_p$ onto $\mathbb{Z}_p$ and this provides that $g \in \mathbb{Z}_p[s]$. Case $n = m$: Since lower terms vanish, we obtain a constant term $\nabla_h^m f(s) = \nabla_h^m a_m s^m = h^m m!\, a_m \in \mathbb{Z}_p$. For $h = 1$ this implies $m!\, a_m \in \mathbb{Z}_p$. Case $n > m$: The constant terms vanish. $\square$

Lemma 2.7. *Let h, m, n be positive integers with $m \geq n$. Then*

$$\nabla_h^n \binom{s}{m} = \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h \binom{s+\nu}{m-n}, \quad s \in \mathbb{Z}_p,$$

where the h-nomial coefficients of order n coincide with

$$(1 + x + \cdots + x^{h-1})^n = \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h x^\nu \quad \textit{and} \quad \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h = h^n. \tag{2.1}$$

Proof. Let $s \in \mathbb{Z}_p$. For $h = 1$ this gives the usual differences $\nabla^n \binom{s}{m} = \binom{s}{m-n}$ for $m \geq n \geq 1$. Now let $h > 1$. Using $\binom{s+1}{m} = \binom{s}{m} + \binom{s}{m-1}$ successively, we get

$$\nabla_h \binom{s}{m} = \sum_{\nu=0}^{h-1} \binom{s+\nu}{m-1}.$$

Applying the above equation repeatedly yields the result, whereas the coefficients are mapped, step by step for $r = 1, \ldots, n$, in the same way as $(1 + x + \cdots + x^{h-1})^{r-1} \mapsto (1 + x + \cdots + x^{h-1})^r$. Then taking $x = 1$ shows that the sum of the h-nomial coefficients of order n equals h^n. $\square$

Proposition 2.8. *Let h, k, n be integers with $h, n \geq 1$ and $k \geq 0$. Then*

$$\theta(n,k) = h^{-n} p^k \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h \binom{\nu}{k} \in \mathbb{Z}_p.$$

Proof. Note that $\theta(n,k) = 0$ for $k > n(h-1)$ and $\theta(n,0) = 1$ by (2.1). According to Lemma 2.7 set $f(x) = 1 + x + \cdots + x^{h-1}$. We will evaluate formal derivatives of both sides of (2.1). Define the differential operator $\mathcal{D}^r = (d/dx)^r / r!$ for $r \geq 0$. We firstly have

$$\mathcal{D}^r f(x) \Big|_{x=1} = \sum_{\nu=r}^{h-1} \binom{\nu}{r} = \binom{h}{r+1},$$

which is valid for all $r \geq 0$. Let $k \geq 1$. We then deduce that

$$\begin{aligned}\mathcal{D}^k f(x)^n \Big|_{x=1} &= \sum_{\nu_1+\cdots+\nu_n=k} \frac{1}{k!}\binom{k}{\nu_1,\ldots,\nu_n} f^{(\nu_1)}(x)\cdots f^{(\nu_n)}(x)\Big|_{x=1} \\ &= \sum_{\nu_1+\cdots+\nu_n=k} \binom{h}{\nu_1+1}\cdots\binom{h}{\nu_n+1} \\ &= \sum_{\nu_1+\cdots+\nu_n=k} \frac{h^n}{(\nu_1+1)\cdots(\nu_n+1)}\binom{h-1}{\nu_1}\cdots\binom{h-1}{\nu_n}. \end{aligned} \tag{2.2}$$

Since $p^l/(l+1) \in \mathbb{Z}_p$ for $l \geq 0$, we obtain for $\nu_1+\cdots+\nu_n = k$ that

$$\frac{p^k}{(\nu_1+1)\cdots(\nu_n+1)} = \frac{p^{\nu_1}}{\nu_1+1}\cdots\frac{p^{\nu_n}}{\nu_n+1} \in \mathbb{Z}_p. \tag{2.3}$$

Thus, replacing h^n by p^k in (2.2) and using (2.3), this shows that

$$h^{-n}p^k\,\mathcal{D}^k f(x)^n\Big|_{x=1} \in \mathbb{Z}_p.$$

Now, consider the right hand side of (2.1). We finally achieve that

$$h^{-n}p^k\,\mathcal{D}^k f(x)^n\Big|_{x=1} = h^{-n}p^k \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h \binom{\nu}{k} \in \mathbb{Z}_p. \qquad \square$$

Proposition 2.9. *Let h, m, n be positive integers. Then*

$$f_n(s) = h^{-n}p^{m-n} \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h \binom{s+\nu}{m-n} \in \mathbb{Z}_p[s]$$

for $n = 1, \ldots, m$.

Proof. We use the Vandermonde's convolution identity, cf. [13, Ch. 5.1, p. 170], that

$$\binom{s+a}{n} = \sum_{k=0}^{n} \binom{a}{k}\binom{s}{n-k}.$$

Hence

$$\begin{aligned} f_n(s) &= h^{-n}p^{m-n} \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h \binom{s+\nu}{m-n} \\ &= h^{-n}p^{m-n} \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h \sum_{k=0}^{m-n} \binom{\nu}{k}\binom{s}{m-n-k} \\ &= \sum_{k=0}^{m-n} p^{m-n-k}\binom{s}{m-n-k}\, h^{-n}p^k \sum_{\nu=0}^{n(h-1)} \binom{n}{\nu}_h \binom{\nu}{k}. \end{aligned}$$

Lemma 2.5 and Proposition 2.8 provide that $f_n \in \mathbb{Z}_p[s]$. $\square$

Proof of Theorem 2.4. Using Lemma 2.7 and Proposition 2.9, we obtain

$$\operatorname{ord}_p \left(p^m \, \nabla_h^n \binom{s}{m} \right) = \operatorname{ord}_p \left((ph)^n \, f_n(s) \right) \geq n(1 + \operatorname{ord}_p h). \qquad \square$$

3. p-adic Kummer spaces

Definition 3.1. *We introduce the spaces $\mathcal{K}_{p,1}$, $\mathcal{K}_{p,2}$, and $\mathcal{K}_{p,2}^{\sharp}$, which we call p-adic Kummer spaces. Furthermore we define the sets $\widehat{\mathcal{K}}_{p,1}$ and $\widehat{\mathcal{K}}_{p,2}$. We distinguish between the following congruences of a function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ for any $s, t \in \mathbb{Z}_p$, $s \neq t$, and any $n \geq 0$:*

(1) ***Kummer congruences**: If $f \in \mathcal{K}_{p,1}$, then*

$$s \equiv t \pmod{p^n \mathbb{Z}_p} \quad \Longrightarrow \quad f(s) \equiv f(t) \pmod{p^{n+1} \mathbb{Z}_p}.$$

If the converse also holds, then $f \in \widehat{\mathcal{K}}_{p,1}$.

(2) ***Kummer type congruences I**: If $f \in \mathcal{K}_{p,2}$, then*

$$\nabla^n f(s) \equiv 0 \pmod{p^n \mathbb{Z}_p}.$$

We write $\Delta_f(n) = \nabla^n f(0)/p^n$, where $\Delta_f(n) \in \mathbb{Z}_p$. Furthermore we write

$$\Delta_f \equiv \Delta_f(1) \pmod{p\mathbb{Z}_p}, \quad 0 \leq \Delta_f < p.$$

If $\Delta_f \neq 0$, additionally $2 \mid \Delta_f(2)$ in case $p = 2$, then $f \in \widehat{\mathcal{K}}_{p,2}$.

(3) ***Kummer type congruences II**: If $f \in \mathcal{K}_{p,2}^{\sharp}$, then*

$$\nabla_h^n f(s) \equiv 0 \pmod{p^{nr} \mathbb{Z}_p}$$

for any $h \geq 1$, where $r = 1 + \operatorname{ord}_p h$.

By definition we have $\widehat{\mathcal{K}}_{p,1} \subset \mathcal{K}_{p,1}$, $\widehat{\mathcal{K}}_{p,2} \subset \mathcal{K}_{p,2}$, and $\mathcal{K}_{p,2}^{\sharp} \subset \mathcal{K}_{p,2}$. Clearly, a function $f \in \mathcal{K}_{p,\nu}$, $\nu = 1, 2$, resp., $f \in \mathcal{K}_{p,2}^{\sharp}$ is continuous on $\mathbb{Z}_p$. For $f \in \mathcal{K}_{p,1}$ this follows by the usual ε-δ criterion of continuity. For $f \in \mathcal{K}_{p,2}$, resp., $f \in \mathcal{K}_{p,2}^{\sharp}$ this is a consequence of its Mahler expansion. The definition of $\mathcal{K}_{p,2}$ above can be weakened as follows.

Lemma 3.2. *Let the function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ satisfy $\nabla^n f(0) \in p^n \mathbb{Z}_p$ for all $n \geq 0$. Then $f \in \mathcal{K}_{p,2}$.*

Proof. The Mahler expansion of f is easily given by

$$f(s) = \sum_{\nu \geq 0} \Delta_f(\nu) \, p^\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p. \tag{3.1}$$

Since $\nabla^n \, p^\nu \binom{s}{\nu} \equiv 0 \pmod{p^n \mathbb{Z}_p}$ for all $n, \nu \geq 0$, it follows that $\nabla^n f(s) \equiv 0 \pmod{p^n \mathbb{Z}_p}$ for all $n \geq 0$ independently of $s \in \mathbb{Z}_p$. $\square$

Lemma 3.3. *The space $\mathcal{K}_{p,2}$ has the basis*

$$\mathcal{B}_p = \left\{ p^\nu \binom{s}{\nu} \right\}_{\nu \geq 0}.$$

Proof. Clearly, $\nabla^n p^\nu \binom{s}{\nu} \equiv 0 \pmod{p^n \mathbb{Z}_p}$ for all $n, \nu \geq 0$. The functions of $\mathcal{B}_p$ are linearly independent over $\mathbb{Z}_p$. Let $f \in \mathcal{K}_{p,2}$. The unique Mahler expansion of f is given as in (3.1). Thus $(\Delta_f(0), \Delta_f(1), \Delta_f(2), \ldots)$ are the coordinates with respect to the basis $\mathcal{B}_p$. Conversely, prescribed values $\Delta_f(\nu) \in \mathbb{Z}_p$ define a uniformly convergent Mahler series, which then equals the Mahler expansion in (3.1). □

Lemma 3.4. *We define the operator $\eth^r = p^{-r} \nabla^r$ on $\mathcal{K}_{p,2}$ for $r \geq 0$. If $f \in \mathcal{K}_{p,2}$, then $g = \eth^r f \in \mathcal{K}_{p,2}$, where we have a shift of coefficients such that $\Delta_g(\nu) = \Delta_f(\nu + r)$ for all $\nu \geq 0$. Moreover, we have the relation $\Delta_f(r) = \eth^r f(0)$.*

Proof. Applying $\eth^r$ to (3.1), this yields

$$\eth^r f(s) = \sum_{\nu \geq 0} \Delta_f(\nu + r)\, p^\nu \binom{s}{\nu} = \sum_{\nu \geq 0} \Delta_g(\nu)\, p^\nu \binom{s}{\nu} = g(s) \in \mathcal{K}_{p,2}.$$

By definition follows that $\Delta_f(r) = \nabla^r f(0)/p^r = \eth^r f(0)$. □

The next theorem shows among other relations that the Kummer type congruences imply the Kummer congruences, but the converse does not hold. We will prove the theorem by the following propositions and corollaries.

Theorem 3.5. *We have the following relations:*

(1) $\mathcal{K}_{p,2} \subsetneq \mathcal{K}_{p,1}$.
(2) $\widehat{\mathcal{K}}_{p,2} \subsetneq \widehat{\mathcal{K}}_{p,1}$.
(3) $\mathcal{K}_{p,2} = \mathcal{K}^\sharp_{p,2}$.
(4) $\mathcal{K}_{p,2} \subset \mathcal{S}^1(\mathbb{Z}_p)$.

Proposition 3.6. *We have $\mathcal{K}_{p,2} = \mathcal{K}^\sharp_{p,2}$.*

Proof. By definition we have $\mathcal{K}^\sharp_{p,2} \subset \mathcal{K}_{p,2}$. Now, we have to show that $\mathcal{K}_{p,2} \subset \mathcal{K}^\sharp_{p,2}$. Let h, n be positive integers and $f \in \mathcal{K}_{p,2}$. Since the sequence $(f_m)_{m \geq 1}$ of the partial sums f_m of the Mahler expansion of f is uniformly convergent to f, we can apply the operator ∇^n_h term by term:

$$\nabla^n_h f(s) = \sum_{\nu \geq 0} \Delta_f(\nu)\, p^\nu\, \nabla^n_h \binom{s}{\nu}.$$

The lower terms for $\nu < n$ vanish by Lemma 2.6, while Theorem 2.4 gives an estimate for the other terms where $\nu \geq n$:

$$\operatorname{ord}_p \left(p^\nu\, \nabla^n_h \binom{s}{\nu} \right) \geq nr, \quad r = 1 + \operatorname{ord}_p h.$$

Thus $\nabla^n_h f(s) \equiv 0 \pmod{p^{nr}}$, which shows that $f \in \mathcal{K}^\sharp_{p,2}$. □

Remark. The fact that $\mathcal{K}_{p,2} = \mathcal{K}^\sharp_{p,2}$ is only caused by properties of binomial polynomials given in Theorem 2.4. These properties ensure that Kummer type congruences I already imply Kummer type congruences II.

Proposition 3.7. *Let $f \in \mathcal{K}_{p,2}$ and $s, t \in \mathbb{Z}_p$, $s \neq t$. We have the following statements:*

(1)
$$\frac{f(s) - f(t)}{s - t} \equiv 0 \pmod{p\mathbb{Z}_p}. \tag{3.2}$$

(2)
$$\frac{f(s) - f(t)}{p\,(s - t)} \equiv \Delta_f \pmod{p\mathbb{Z}_p}. \tag{3.3}$$

(3)
$$f'(s) \equiv p\,\Delta_f \pmod{p^2\mathbb{Z}_p}.$$

For the cases (2) and (3) we additionally require that $2 \mid \Delta_f(2)$ when $p = 2$.

Proof. Since $f \in \mathcal{K}_{p,2}$, we make use of the Mahler expansion of f and show for $s \neq t$ that
$$\frac{\sum_{\nu \geq 0} \Delta_f(\nu)\, p^\nu \left[\binom{s}{\nu} - \binom{t}{\nu}\right]}{p\,(s-t)} = \Delta_f(1) + \Delta_f(2)\frac{p}{2}(s + t - 1) + \mathcal{O}(p). \tag{3.4}$$
The lower terms are easily given. For the higher terms with $\nu \geq 3$, we have the following estimate by Lemma 2.3:
$$\operatorname{ord}_p\left(\binom{s}{\nu} - \binom{t}{\nu}\right) \geq \operatorname{ord}_p(s-t) - \lfloor \log_p \nu \rfloor,$$
where $\log_p$ is the real-valued logarithm to base p. Since $r = \nu - 1 - \lfloor \log_p \nu \rfloor \geq 1$ for $\nu \geq 3$ and all primes p, we obtain in these cases that
$$p^\nu \left(\binom{s}{\nu} - \binom{t}{\nu}\right) \Big/ p\,(s-t) \in p^r \mathbb{Z}_p, \tag{3.5}$$
where $r \to \infty$ as $\nu \to \infty$. Therefore (3.2) follows by (3.4). By definition $\Delta_f(1) \equiv \Delta_f \pmod{p\mathbb{Z}_p}$ and in case $p = 2$ we have $2 \mid \Delta_f(2)$, thus (3.3) follows by (3.4). Note that (3.4) is responsible for the extra condition in the case $p = 2$. Now, taking any sequence $(s_\nu, t_\nu)_{\nu \geq 1} \to (s, s)$ where $s_\nu \neq t_\nu$, (3.4) and (3.5) show the existence of a limit: $f'(s) \equiv p\,\Delta_f \pmod{p^2\mathbb{Z}_p}$. □

Corollary 3.8. *We have $\mathcal{K}_{p,2} \subsetneq \mathcal{K}_{p,1}$ and $\widehat{\mathcal{K}}_{p,2} \subsetneq \widehat{\mathcal{K}}_{p,1}$. If $f \in \widehat{\mathcal{K}}_{p,2}$, then a strong version of the Kummer congruences holds, that*
$$|f(s) - f(t)|_p = |p\,(s-t)|_p\,, \quad s, t \in \mathbb{Z}_p.$$

Proof. Let $f \in \mathcal{K}_{p,2}$ and $s, t \in \mathbb{Z}_p$. We omit the trivial case $s = t$ and assume that $s \neq t$. Eq. (3.4) shows that $|f(s) - f(t)|_p \leq |p\,(s-t)|_p$. This implies the Kummer congruences and that $f \in \mathcal{K}_{p,1}$. For $f \in \widehat{\mathcal{K}}_{p,2}$, we have $\Delta_f \neq 0$ and additionally in case $p = 2$ that $2 \mid \Delta_f(2)$. Then (3.3) yields $|(f(s) - f(t))/(p\,(s-t))|_p = 1$, which gives the equation above and shows that $f \in \widehat{\mathcal{K}}_{p,1}$. It remains to show that $\mathcal{K}_{p,2} \neq \mathcal{K}_{p,1}$ and $\widehat{\mathcal{K}}_{p,2} \neq \widehat{\mathcal{K}}_{p,1}$.

We construct a function $f \in \mathcal{K}_{p,2}$, resp., $f \in \widehat{\mathcal{K}}_{p,2}$ by choosing suitable coefficients $\Delta_f(\nu)$. Therefore we can assume that an index $n \geq 5$ exists, where $\Delta_f(n) \in \mathbb{Z}_p^*$. Now define

$$\tilde{f}(s) = \sum_{\nu \geq 0} \Delta_f(\nu)\, p^{\nu - \delta_{n,\nu}} \binom{s}{\nu}$$

with $\delta_{n,\nu}$ as Kronecker's delta. Then $\tilde{f}$ also satisfies (3.4), since higher terms do not play a role in view of (3.5). This implies that $\tilde{f} \in \mathcal{K}_{p,1}$, resp., $\tilde{f} \in \widehat{\mathcal{K}}_{p,1}$. By construction

$$\nabla^n \tilde{f}(s) \equiv \Delta_f(n)\, p^{n-1} \not\equiv 0 \pmod{p^n \mathbb{Z}_p},$$

consequently $\tilde{f} \notin \mathcal{K}_{p,2}$, resp., $\tilde{f} \notin \widehat{\mathcal{K}}_{p,2}$. □

Corollary 3.9. *We have*

$$\mathcal{K}_{p,2} \subset \mathcal{S}^1(\mathbb{Z}_p) \subset \mathrm{Lip}(\mathbb{Z}_p) \subset \mathcal{C}(\mathbb{Z}_p).$$

If $f \in \mathcal{K}_{p,2}$, then the Volkenborn integral of f is given by

$$\int_{\mathbb{Z}_p} f(s)\, ds = \sum_{\nu \geq 0} (-1)^\nu \frac{\Delta_f(\nu)\, p^\nu}{\nu + 1} \in \mathbb{Z}_p.$$

Proof. As seen in Proposition 3.7, a function $f \in \mathcal{K}_{p,2}$ is Lipschitz and moreover strictly differentiable. The latter is also a consequence that the coefficients of the Mahler expansion obey that $\left|\Delta_f(\nu)\, p^\nu/\nu\right|_p \to 0$ as $\nu \to \infty$, see [26, Ch. 5.1, p. 227]. Therefore $\mathcal{K}_{p,2} \subset \mathcal{S}^1(\mathbb{Z}_p) \subset \mathrm{Lip}(\mathbb{Z}_p) \subset \mathcal{C}(\mathbb{Z}_p)$. As a further consequence that $f \in \mathcal{S}^1(\mathbb{Z}_p)$, the Volkenborn integral is given as above, see [26, Ch. 5.2, p. 265]. Since $p^\nu/(\nu+1)$ is p-integral, the sum lies in $\mathbb{Z}_p$. □

This proves Theorem 3.5. □ Functions of $\mathcal{K}_{p,1}$ and $\mathcal{K}_{p,2}$ obey the Kummer congruences. Moreover, one can easily calculate any values $(\mathrm{mod}\ p^{n+1}\mathbb{Z}_p)$ of functions of $\mathcal{K}_{p,2}$ by a finite Mahler expansion. Additionally, we give a formulation in terms like the Kummer congruences.

Proposition 3.10. *Let $f \in \mathcal{K}_{p,2}$. For $n \geq 0$ and $s \in \mathbb{Z}_p$ we have*

$$f(s) \equiv \sum_{\nu=0}^{n} \Delta_f(\nu)\, p^\nu \binom{s}{\nu} \pmod{p^{n+1}\mathbb{Z}_p},$$

resp.,

$$f(s) \equiv \sum_{\nu=0}^{n} f(\nu) \binom{s}{\nu} \binom{n-s}{n-\nu} \pmod{p^{n+1}\mathbb{Z}_p}.$$

Proof. For $n \geq 0$ we have a finite Mahler expansion

$$f(s) \equiv \sum_{j=0}^{n} \Delta_f(j)\, p^j \binom{s}{j} \equiv \sum_{j=0}^{n} \nabla^j f(0) \binom{s}{j} \pmod{p^{n+1}\mathbb{Z}_p}.$$

By definition we have

$$\nabla^j f(0) = \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} f(k) = \sum_{k=0}^{n} \binom{j}{k} (-1)^{j-k} f(k).$$

We rearrange the finite sums and omit the vanishing terms. Hence

$$f(s) \equiv \sum_{k=0}^{n} f(k) \sum_{j=k}^{n} (-1)^{j-k} \binom{j}{k} \binom{s}{j} \pmod{p^{n+1}\mathbb{Z}_p}.$$

We use the following identities, cf. [13, Ch. 5.1, pp. 164–168], for $j \geq k$:

$$\binom{s}{j}\binom{j}{k} = \binom{s}{k}\binom{s-k}{j-k}$$

and

$$\sum_{j=k}^{n} (-1)^{j-k} \binom{s-k}{j-k} = \sum_{j=0}^{n-k} (-1)^j \binom{s-k}{j} = (-1)^{n-k} \binom{s-k-1}{n-k} = \binom{n-s}{n-k}.$$

This gives the result. □

As a consequence we obtain a kind of a reflection formula.

Corollary 3.11. *Let $f \in \mathcal{K}_{p,2}$, $n \geq 0$, and $s \in \mathbb{Z}_p$. Then there exist coefficients $a_\nu \in \mathbb{Z}_p$ depending on s such that*

$$f(s) \equiv \sum_{\nu=0}^{n} a_\nu \, f(\nu) \qquad \pmod{p^{n+1}\mathbb{Z}_p},$$

$$f(n-s) \equiv \sum_{\nu=0}^{n} a_{n-\nu} \, f(\nu) \pmod{p^{n+1}\mathbb{Z}_p}.$$

Proof. Set $a_\nu = \binom{s}{\nu}\binom{n-s}{n-\nu}$ for $\nu = 0, \ldots, n$. □

As a further consequence, we get an expression via differences for values at negative integer arguments.

Corollary 3.12. *Let $f \in \mathcal{K}_{p,2}$. Let n, r be integers, where $n \geq 0$ and $r > 0$. Then*

$$f(-r) \equiv (-1)^n \, r \binom{n+r}{r} \nabla^n \frac{f(s)}{s+r} \Big|_{s=0} \pmod{p^{n+1}\mathbb{Z}_p}.$$

Proof. From Proposition 3.10 we have

$$f(-r) \equiv \sum_{\nu=0}^{n} f(\nu) \binom{-r}{\nu} \binom{n+r}{n-\nu} \pmod{p^{n+1}\mathbb{Z}_p}.$$

The result follows by

$$\binom{-r}{\nu}\binom{n+r}{n-\nu} = (-1)^\nu \binom{\nu+r-1}{r-1}\binom{n+r}{\nu+r} = (-1)^\nu \binom{n}{\nu}\binom{n+r}{r} \frac{r}{\nu+r}.$$ □

Lemma 3.13. *Let $f \in \mathcal{K}_{p,2}$ and $\lambda(s) = a+bs$, where $a, s \in \mathbb{Z}_p$ and $b \in \mathbb{Z}$. Then $f \circ \lambda \in \mathcal{K}_{p,2}$, where $(f \circ \lambda)(s) = f(\lambda(s))$. Moreover, if $b \neq 0$ then $\nabla^n (f \circ \lambda)(s) \equiv 0 \pmod{p^{nr}}$, where $r = 1 + \operatorname{ord}_p b$.*

Proof. Case $b = 0$: This gives a constant function $f(a) \in \mathcal{K}_{p,2}$. Case $b > 0$: We have $\lambda(s+\nu) = a + bs + b\nu$. Since $\mathcal{K}_{p,2}^\sharp = \mathcal{K}_{p,2}$, we obtain

$$\nabla^n (f \circ \lambda)(s) = \nabla_b^n f(t)\,\Big|_{t=a+bs} \equiv 0 \pmod{p^{nr}}$$

for all $n \geq 1$, where $r = 1 + \operatorname{ord}_p b$. Thus $f \circ \lambda \in \mathcal{K}_{p,2}$. Case $b < 0$: Set $b' = |b|$, then $\lambda(s+\nu) = a - b's - b'\nu$. Recall Definition 2.1 and note the symmetry of the binomial coefficients. We then get

$$\nabla^n (f \circ \lambda)(s) = (-1)^n \, \nabla_{b'}^n f(t)\,\Big|_{t=a-b's-b'n} \equiv 0 \pmod{p^{nr}}$$

for all $n \geq 1$, where $r = 1 + \operatorname{ord}_p b' = 1 + \operatorname{ord}_p b$. Consequently $f \circ \lambda \in \mathcal{K}_{p,2}$. $\square$

4. Zeros and fixed points

Definition 4.1. *Let $f \in \mathcal{C}(\mathbb{Z}_p)$ and $n \geq 0$. We call*

$$f_n(s) = p^{-n} f(\xi_n + s\,p^n), \quad s \in \mathbb{Z}_p,$$

a function of level n of f, when f_n defines a function $f_n : \mathbb{Z}_p \to \mathbb{Z}_p$, where the p-adic expansion of ξ_n is given by

$$\xi_0 = 0, \quad \xi_n = s_0 + s_1\,p + \cdots + s_{n-1}\,p^{n-1} \quad \textit{for } n \geq 1.$$

Proposition 4.2. *Let $f \in \operatorname{Lip}(\mathbb{Z}_p)$ satisfy*

$$\frac{f(s) - f(t)}{s - t} \equiv \Delta \pmod{p\mathbb{Z}_p} \quad \textit{for } s \neq t, \quad s, t \in \mathbb{Z}_p,$$

such that Δ is a fixed integer where $0 < \Delta < p$. Then f has a unique simple zero $\xi \in \mathbb{Z}_p$ and

$$f(s) = (s - \xi)\, f^*(s), \quad |f(s)|_p = |s - \xi|_p\,, \quad s \in \mathbb{Z}_p,$$

where $f^*(s) \equiv \Delta \pmod{p\mathbb{Z}_p}$ and $f^* \in \mathcal{C}(\mathbb{Z}_p)^*$.

Proof. We show on induction that there exists a sequence $(f_n)_{n\geq 0}$ of functions of level n of f, such that the sequence $(\xi_n)_{n\geq 0}$ is uniquely determined. Basis of induction $n = 0$: $f_0(s) = f(s)$. Inductive step $n \mapsto n+1$: Assume this is true for n prove for $n+1$. We have

$$\begin{aligned} \nabla f_n(s) &= p^{-n} \left(f(\xi_n + (s+1)\,p^n) - f(\xi_n + s\,p^n) \right) \\ &= \frac{f(\xi_n + (s+1)\,p^n) - f(\xi_n + s\,p^n)}{(\xi_n + (s+1)\,p^n) - (\xi_n + s\,p^n)}. \end{aligned} \tag{4.1}$$

By assumption we get $\nabla f_n(s) \equiv \Delta \not\equiv 0 \pmod{p\mathbb{Z}_p}$. Thus, we can uniquely determine the value s_n by

$$s_n \equiv -f_n(0)/\Delta \pmod{p\mathbb{Z}_p}, \quad 0 \leq s_n < p, \tag{4.2}$$

such that $f_n(s_n) \equiv 0 \pmod{p\mathbb{Z}_p}$. It also follows that

$$f_n(s_n + s\,p) \equiv 0 \pmod{p\mathbb{Z}_p} \quad \text{for } s \in \mathbb{Z}_p.$$

Setting $\xi_{n+1} = \xi_n + s_n\, p^n$, we obtain the function $f_{n+1} : \mathbb{Z}_p \to \mathbb{Z}_p$ by

$$f_{n+1}(s) = p^{-1}\, f_n(s_n + s\,p).$$

Existence of the zero: We achieve that $\lim_{n\to\infty} |f(\xi_n)|_p = 0$. Define $\xi = \lim_{n\to\infty} \xi_n$, then ξ is a zero of f, due to the fact that $f \in \mathrm{Lip}(\mathbb{Z}_p) \subset \mathcal{C}(\mathbb{Z}_p)$. Uniqueness of the zero: Assume to the contrary that ξ and ξ' are different zeros of f. Then

$$0 = \frac{f(\xi) - f(\xi')}{\xi - \xi'} \equiv \Delta \not\equiv 0 \pmod{p\mathbb{Z}_p}$$

yields a contradiction. Representation of f: Since $f(\xi) = 0$, we obtain

$$f^*(s) = \frac{f(s)}{s - \xi} \equiv \Delta \pmod{p\mathbb{Z}_p} \quad \text{for } s \neq \xi, \quad s \in \mathbb{Z}_p.$$

We get $\lim_{s\to\xi} f^*(s) = f'(\xi)$ where $f^*(\xi) = f'(\xi) \equiv \Delta \pmod{p\mathbb{Z}_p}$. This implies that $f^* \in \mathcal{C}(\mathbb{Z}_p)$. Since $1/f^* \in \mathcal{C}(\mathbb{Z}_p)$, we even have $f^* \in \mathcal{C}(\mathbb{Z}_p)^*$. Finally, $f(s) = (s - \xi)\, f^*(s)$ and $|f(s)|_p = |s - \xi|_p$ for $s \in \mathbb{Z}_p$. □

Remark. This result is similar to Hensel's Lemma, which predicts a zero ξ of a polynomial $g \in \mathbb{Z}_p[s]$, when $|g(\tilde{s})|_p < |g'(\tilde{s})|_p^2$ for some $\tilde{s} \in \mathbb{Z}_p$. Then $|\xi - \tilde{s}|_p = |g(\tilde{s})/g'(\tilde{s})|_p$, cf. [26, Ch. 2.1, p. 80]. But in this context, a function $f \in \mathrm{Lip}(\mathbb{Z}_p)$, that satisfies the conditions of Proposition 4.2, can have an infinite Mahler expansion in view of Proposition 3.7. Moreover this function has only one zero. Note also that this result cannot be derived by the p-adic Weierstrass Preparation Theorem, cf. [29, Thm. 7.3, p. 115], since $s - \xi$ is not a distinguished polynomial when $\xi \in \mathbb{Z}_p^*$.

Proposition 4.3. *Let $f \in \mathrm{Lip}(\mathbb{Z}_p)$ satisfy*

$$\frac{f(s) - f(t)}{s - t} \equiv 0 \pmod{p\mathbb{Z}_p} \quad \textit{for } s \neq t, \quad s, t \in \mathbb{Z}_p.$$

We have the following statements:

1. *The function f has a fixed point $\tau \in \mathbb{Z}_p$.*
2. *If there exists a $\xi \in \mathbb{Z}_p$ such that $f(\xi) \in p^n\mathbb{Z}_p$ with $n \geq 1$, then there exists a function f_n of level n where $\xi_n \equiv \xi \pmod{p^n\mathbb{Z}_p}$ and $f_n(s) \equiv p^{-n} f(\xi) \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$.*

Proof. (1): Since $(\mathbb{Z}_p, |\cdot|_p)$ is a Banach space and the function f defines a contractive mapping by $|f(s) - f(t)|_p \leq p^{-1}\,|s - t|_p$ for $s \neq t$, the Banach fixed point theorem provides a $\tau \in \mathbb{Z}_p$ such that $f(\tau) = \tau$. (2): We have $|f(s) - f(t)|_p \leq |p(s - t)|_p$, which implies the Kummer congruences. Assume that $f(\xi) \in p^n\mathbb{Z}_p$. According to Definition 4.1, we define f_n and determine $\xi_n \in \mathbb{Z}_p$ by $\xi_n \equiv \xi \pmod{p^n\mathbb{Z}_p}$. Then we get $f(\xi_n) \equiv f(\xi) \pmod{p^{n+1}\mathbb{Z}_p}$. Similar to (4.1) we obtain $\nabla f_n(s) \equiv 0 \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$. Consequently, $f_n(s) \equiv p^{-n} f(\xi) \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$. □

Definition 4.4. *We define for $\mathcal{T} = \mathcal{K}_{p,\nu}, \widehat{\mathcal{K}}_{p,\nu}$, $\nu = 1, 2$, the decomposition $\mathcal{T} = \mathcal{T}^0 \cup \mathcal{T}^*$ where*

$$\mathcal{T}^0 = \{f \in \mathcal{T} : f(0) \in p\mathbb{Z}_p\}, \quad \mathcal{T}^* = \{f \in \mathcal{T} : f(0) \in \mathbb{Z}_p^*\}.$$

Theorem 4.5. *We have the following statements:*

(1) *If $f \in \mathcal{K}_{p,1}$ or $f \in \mathcal{K}_{p,2}$, then f has a fixed point $\tau \in \mathbb{Z}_p$.*
(2) *If $f \in \mathcal{K}_{p,1}^*$ or $f \in \mathcal{K}_{p,2}^*$, then $|f(s)|_p = 1$ for $s \in \mathbb{Z}_p$.*
(3) *If $f \in \widehat{\mathcal{K}}_{p,2}^0$, then f has a unique simple zero $\xi \in \mathbb{Z}_p$ and*

$$f(s) = p\,(s - \xi)\, f^*(s), \quad |f(s)|_p = |p\,(s - \xi)|_p\,, \quad s \in \mathbb{Z}_p,$$

where $f^*(s) \equiv \Delta_f \pmod{p\mathbb{Z}_p}$ and $f^* \in \mathcal{C}(\mathbb{Z}_p)^*$.

Proof. (1): Case $f \in \mathcal{K}_{p,1}$: Since $|f(s) - f(t)|_p \leq |p(s-t)|_p$ for $s \neq t$, this shows the existence of a fixed point as argued in the proof of Proposition 4.3. Case $f \in \mathcal{K}_{p,2}$: Proposition 3.7 provides that f satisfies the condition of Proposition 4.3. (2): Clearly, since $f(0) \equiv f(s) \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$. (3): Proposition 3.7 shows that f/p satisfies the conditions of Proposition 4.2. □

Functions $f \in \widehat{\mathcal{K}}_{p,2}^0$ play a significant role as we will later see in the next sections. Another characterization of $\widehat{\mathcal{K}}_{p,2}^0$ is given by

$$\widehat{\mathcal{K}}_{p,2}^0 = \left\{ f \in \mathcal{K}_{p,2} : f \notin \mathcal{K}_{p,2}^*, \eth f \in \mathcal{K}_{p,2}^* \right\}.$$

The following algorithm shows how to compute an approximation $\pmod{p^n\mathbb{Z}_p}$ of the zero of a function $f \in \widehat{\mathcal{K}}_{p,2}^0$. For this task we need the values

$$f(0)/p, \ldots, f(n)/p \pmod{p^n\mathbb{Z}_p}.$$

We present two possible methods: The first uses the values of f, the second its Mahler coefficients.

Algorithm 4.6. *Let $f \in \widehat{\mathcal{K}}_{p,2}^0$ and ξ be the unique zero of f. Let $n \geq 1$ be fixed. Initially set $\xi_0 = 0$ and $\delta \equiv -\Delta_f^{-1} \pmod{p\mathbb{Z}_p}$. Further compute for $\nu = 0, \ldots, n$ the values*

$$\tilde{f}_\nu \equiv f(\nu)/p \pmod{p^n\mathbb{Z}_p},$$

resp.,

$$\Delta_f(\nu) \pmod{p^{n-\nu+1}\mathbb{Z}_p}.$$

For each step $r = 1, \ldots, n$ proceed as follows. Compute

$$\gamma_{r-1} \equiv \sum_{\nu=0}^{r} \tilde{f}_\nu \binom{\xi_{r-1}}{\nu} \binom{r - \xi_{r-1}}{r - \nu} \pmod{p^r\mathbb{Z}_p},$$

resp.,

$$\gamma_{r-1} \equiv \sum_{\nu=0}^{r} \Delta_f(\nu)\, p^{\nu-1} \binom{\xi_{r-1}}{\nu} \pmod{p^r\mathbb{Z}_p}. \tag{4.3}$$

Then $\gamma_{r-1} \in p^{r-1}\mathbb{Z}_p$. Set $s_{r-1} \equiv \gamma_{r-1}\delta/p^{r-1} \pmod{p\mathbb{Z}_p}$ where $0 \leq s_{r-1} < p$. Set $\xi_r = \xi_{r-1} + s_{r-1}\, p^{r-1}$ and go to the next step while $r < n$. Finally, $\xi_n \equiv \xi \pmod{p^n\mathbb{Z}_p}$.

Proof. The function f/p satisfies the conditions of Proposition 4.2. We have to adapt the procedure given there to compute the zero of f/p. For each step we use (4.2) to get the next digit of the p-adic expansion of ξ; to compute the term $f_{r-1}(0)$ we make use of Proposition 3.10 modified for f/p. Note that $\Delta_f(0) = f(0) \in p\mathbb{Z}_p$, so (4.3) is valid. □

Remark 4.7. The second method can be further optimized. The binomial coefficient $\binom{\xi_{r-1}}{\nu}$ can be effectively computed $\pmod{p^{r-\nu+1}\mathbb{Z}_p}$, since we already know the p-adic expansion of ξ_{r-1}. For the last term of the sum of (4.3) we can apply the well known theorem of Lucas:

$$\binom{\xi_{r-1}}{r} \equiv \binom{s_0}{r_0} \cdots \binom{s_{r-2}}{r_{r-2}} \pmod{p\mathbb{Z}_p},$$

where r_ν are the digits of the p-adic expansion of r. Let $l = \lfloor \log_p r \rfloor$, then

$$\Delta_f(r)\, p^{r-1} \binom{\xi_{r-1}}{r} \equiv \Delta_f(r)\, p^{r-1} \binom{s_0}{r_0} \cdots \binom{s_l}{r_l} \pmod{p^r\mathbb{Z}_p}.$$

Lucas' theorem can be extended to higher prime powers. Davis and Webb [6] showed a similar formula to compute binomial coefficients modulo p^m, $m \geq 2$, which uses slightly modified binomial coefficients that are evaluated on blocks of m digits.

In the next algorithm, which computes an approximation $\pmod{p^n\mathbb{Z}_p}$ of the fixed point of a function $f \in \mathcal{K}_{p,2}$, we can apply Lucas' theorem.

Algorithm 4.8. *Let $f \in \mathcal{K}_{p,2}$ and τ be the fixed point of f. Let $n \geq 1$ be fixed. Initially set $\tau_1 = t_0 \equiv f(0) \pmod{p\mathbb{Z}_p}$, where $0 \leq t_0 < p$. Further compute the values*

$$\Delta_f(\nu) \pmod{p^{n-\nu}\mathbb{Z}_p}, \quad \nu = 0, \ldots, n-1.$$

For each step $r = 1, \ldots, n-1$ proceed as follows. Compute

$$t_r \equiv p^{-r} \left(\sum_{\nu=0}^{r-1} \Delta_f(\nu)\, p^\nu \binom{\tau_r}{\nu} - \tau_r \right) + \Delta_f(r) \binom{t_0}{r_0} \cdots \binom{t_l}{r_l} \pmod{p\mathbb{Z}_p},$$

where $0 \leq t_r < p$, $l = \lfloor \log_p r \rfloor$, and r_ν are the p-adic digits of r. Set $\tau_{r+1} = \tau_r + t_r\, p^r$ and go to the next step while $r < n-1$. Finally, $\tau_n \equiv \tau \pmod{p^n\mathbb{Z}_p}$.

Proof. By Theorem 4.5 f has a fixed point τ, which solves simultaneously the congruences

$$\tau \equiv \sum_{\nu=0}^{r-1} \Delta_f(\nu)\, p^\nu \binom{\tau}{\nu} \pmod{p^r\mathbb{Z}_p}, \quad r \geq 1.$$

Let $\tau_r = t_0 + t_1\, p + \cdots + t_{r-1}\, p^{r-1}$ be the truncated p-adic expansion of τ for $r \geq 1$ and set $\tau_0 = 0$. Using Lemma 2.5, we observe that

$$p^\nu \binom{\tau_r}{\nu} \equiv \frac{p^\nu}{\nu!}\, (\tau_{r-1})_\nu \pmod{p^r\mathbb{Z}_p}, \quad \nu \geq 0, r \geq 1. \tag{4.4}$$

Thus

$$\tau_r \equiv \sum_{\nu=0}^{r-1} \Delta_f(\nu)\, p^\nu \binom{\tau_{r-1}}{\nu} \pmod{p^r\mathbb{Z}_p}, \quad r \geq 1. \tag{4.5}$$

Now we use induction on r to compute t_r. Basis of induction $r = 1$: By (4.5) we get $\tau_1 = t_0 \equiv f(0) \pmod{p\mathbb{Z}_p}$. Inductive step $r \mapsto r+1$: Assume this is true for r prove for $r+1$. We can use (4.5) for $r+1$ to obtain

$$\tau_{r+1} = \tau_r + t_r\, p^r \equiv \sum_{\nu=0}^{r-1} \Delta_f(\nu)\, p^\nu \binom{\tau_r}{\nu} + \Delta_f(r)\, p^r \binom{\tau_r}{r} \pmod{p^{r+1}\mathbb{Z}_p}.$$

Considering Remark 4.7 and using (4.4) and (4.5), this yields

$$t_r\, p^r \equiv \underbrace{\sum_{\nu=0}^{r-1} \Delta_f(\nu)\, p^\nu \binom{\tau_r}{\nu} - \tau_r}_{\in\, p^r\mathbb{Z}_p} + \Delta_f(r)\, p^r \binom{t_0}{r_0} \cdots \binom{t_l}{r_l} \pmod{p^{r+1}\mathbb{Z}_p},$$

where $l = \lfloor \log_p r \rfloor$. Dividing the above congruence by p^r gives the result. $\square$

We have a close relation between the zero and the fixed point of $f \in \widehat{\mathcal{K}}^0_{p,2}$.

Lemma 4.9. *If $f \in \widehat{\mathcal{K}}^0_{p,2}$ with $f(0) \neq 0$, then $\operatorname{ord}_p f(0) = \operatorname{ord}_p \tau = 1 + \operatorname{ord}_p \xi$. More precisely,*

$$\tau / p\,\xi \equiv -\Delta_f \pmod{p\mathbb{Z}_p} \quad \text{and} \quad f(0)/\tau \equiv 1 \pmod{p\mathbb{Z}_p},$$

where τ is the fixed point and ξ is the zero of f.

Proof. The case $f(0) = 0$ implies that $\xi = \tau = 0$ and vice versa, which we have excluded. Using (3.3) of Proposition 3.7 yields that

$$\frac{\tau}{p\,(\tau - \xi)} \equiv \Delta_f \pmod{p\mathbb{Z}_p}.$$

Since $\Delta_f \neq 0$, we can invert the congruence such that

$$\Delta_f^{-1} \equiv \frac{p\,(\tau - \xi)}{\tau} \equiv -p\,\xi/\tau \pmod{p\mathbb{Z}_p}.$$

This shows the claimed congruence and $|\tau|_p = |p\,\xi|_p$. Furthermore Theorem 4.5 shows that $|f(0)|_p = |p\,\xi|_p$ and also $f(0)/p\,\xi \equiv -\Delta_f \pmod{p\mathbb{Z}_p}$. Thus $f(0)/\tau \equiv 1 \pmod{p\mathbb{Z}_p}$. $\square$

Now, we revisit Theorem 4.5 to show that one cannot improve the result as follows.

Proposition 4.10. *If $f \in \widehat{\mathcal{K}}^0_{p,2}$, then f can be decomposed as*

$$f(s) = p\,(s - \xi)\, f^*(s), \quad s \in \mathbb{Z}_p,$$

with $\xi \in \mathbb{Z}_p$, $f^*(s) \equiv \Delta_f \pmod{p\mathbb{Z}_p}$, and $f^* \in \mathcal{C}(\mathbb{Z}_p)^*$. But in general $f^* \notin \mathcal{K}^*_{p,2}$.

Proof. By Theorem 4.5 we have the decomposition of f as above. We construct the following functions for $p \geq 3$ using the binomial expansion in $\mathbb{Z}_p$, cf. Proposition 6.2 later.

$$f_p(s) = (1+p)^s - 1 = \sum_{\nu \geq 1} p^\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p.$$

Clearly, $f_p \in \widehat{\mathcal{K}}_{p,2}^0$ with $\Delta_{f_p} = 1$ and f_p has a zero $\xi = 0$. Set $f_p^*(s) = f_p(s)/ps$. We easily obtain

$$f_p^*(s) = 1 + \sum_{\nu \geq 1} \frac{p^\nu}{\nu+1} \binom{s-1}{\nu}, \quad s \in \mathbb{Z}_p.$$

It follows that $\nabla^{p-1} f_p^*(1)/p^{p-1} = \frac{1}{p}$ and consequently $f_p^* \notin \mathcal{K}_{p,2}^*$. Now, we consider the remaining case $p = 2$. We have to modify f_p in the following way:

$$\tilde{f}_p(s) = (1+p)^s + p^2 \binom{s}{2} - 1 = \sum_{\nu \geq 1} (1 + \delta_{2,\nu}) p^\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p.$$

Then $\tilde{f}_p$ has the properties $\Delta_{\tilde{f}_p} = 1$ and $2 \mid \Delta_{\tilde{f}_p}(2)$. Thus $\tilde{f}_p \in \widehat{\mathcal{K}}_{p,2}^0$ and $\xi = 0$. Similarly set $\tilde{f}_p^*(s) = \tilde{f}_p(s)/ps$ and as usual $q = 2p$. We derive in this case that $\nabla^{q-1} \tilde{f}_p^*(1)/p^{q-1} = \frac{1}{q}$ and finally $\tilde{f}_p^* \notin \mathcal{K}_{p,2}^*$. $\square$

The proof above works with functions $f \in \mathcal{K}_{p,2}$, that have a zero at $s = 0$. These functions have the following property.

Lemma 4.11. *Let $f \in \mathcal{K}_{p,2}$ with $f(0) = 0$. Then $f(s) = ps\, g(s)$ for $s \in \mathbb{Z}_p$, where $g \in \mathcal{C}(\mathbb{Z}_p)$ and*

$$g(0) = \int_{\mathbb{Z}_p} \eth f(s)\, ds = \sum_{\nu \geq 0} (-1)^\nu \frac{\Delta_f(\nu+1)\, p^\nu}{\nu+1}. \tag{4.6}$$

Proof. Using the Mahler expansion and shifting the index, we obtain

$$g(s) = f(s)/ps = \sum_{\nu > 0} \frac{\Delta_f(\nu)\, p^\nu \binom{s}{\nu}}{ps} = \sum_{\nu \geq 0} \frac{\Delta_f(\nu+1)\, p^\nu}{\nu+1} \binom{s-1}{\nu}. \tag{4.7}$$

Since $p^\nu/(\nu+1) \to 0$ as $\nu \to \infty$ and $p^\nu/(\nu+1)$ is p-integral, we deduce that $g \in \mathcal{C}(\mathbb{Z}_p)$. Comparing the value of $g(0)$ and the Volkenborn integral of $\eth f$ by using Lemma 3.4 and Corollary 3.9 gives the result. $\square$

5. Degenerate functions

Comparing the spaces $\mathcal{K}_{p,1}$ and $\mathcal{K}_{p,2}$, a function $f \in \mathcal{K}_{p,2}$ obeys a very strong law regarding its Mahler expansion. We can think of degenerate functions as follows.

Definition 5.1. *We call a function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ δ-degenerate, if f has a Mahler series such that*

$$f(s) = \sum_{\nu \geq 0} \Delta'_{f,\delta}(\nu)\, p^{\delta(\nu)} \binom{s}{\nu}, \quad s \in \mathbb{Z}_p,$$

where $\Delta'_{f,\delta}(\nu) \in \mathbb{Z}_p$ *and* $\delta : \mathbb{N}_0 \to \mathbb{N}_0$ *is a monotonically increasing function with* $\delta(\nu) \to \infty$ *as* $\nu \to \infty$*. We further define*

$$ {}_{f,\delta}(n) = \min_{\nu \geq 0}\{\nu : \delta(\nu) \geq n\}, $$
$$ \vartheta_{f,\delta} = \min_{\nu \geq 0}\{\nu : \delta(\nu) < \nu\}. $$

The parameter $\vartheta_{f,\delta}$ determines the first index, where f has a *defect* compared to a Mahler expansion of a function of $\mathcal{K}_{p,2}$. Note that δ, depending on f, is not uniquely defined and has to be chosen suitably. This has the following reason. Demanding that $\Delta'_{f,\delta}(\nu) \in \mathbb{Z}_p^*$, we possibly obtain a non-monotonically increasing function δ, which is difficult to handle. Now, we have the following properties and weaker conditions, where we can adapt some results of the previous sections.

Proposition 5.2. *Let f be a δ-degenerate function. We have the following statements:*

(1) *If* $\vartheta_{f,\delta} = \infty$*, then* $f \in \mathcal{K}_{p,2}$.

(2) *If* $\vartheta_{f,\delta} < \infty$*, then*

$$ f(s) = \sum_{\nu=0}^{\vartheta_{f,\delta}-1} \Delta_f(\nu)\, p^\nu \binom{s}{\nu} + \sum_{\nu \geq \vartheta_{f,\delta}} \Delta'_{f,\delta}(\nu)\, p^{\delta(\nu)} \binom{s}{\nu}, \quad s \in \mathbb{Z}_p. $$

(3) *If* $\vartheta_{f,\delta} \geq 3$ *and* $\delta(\nu) \geq 2 + \lfloor \log_p \nu \rfloor$ *for* $\nu \geq 3$*, then* $f \in \mathcal{K}_{p,1}$*. Moreover, we have for* $s \neq t$*,* $s, t \in \mathbb{Z}_p$*, that*

$$ \frac{f(s) - f(t)}{s - t} \equiv 0 \pmod{p\mathbb{Z}_p} $$

and

$$ \frac{f(s) - f(t)}{p\,(s - t)} \equiv \Delta_f \pmod{p\mathbb{Z}_p}, $$

where in the latter case we additionally require that $2 \mid \Delta_f(2)$ *when* $p = 2$.

Proof. (1): Clearly by definition of $\mathcal{K}_{p,2}$. (2): This follows by comparing the Mahler expansion of f up to index $\vartheta_{f,\delta} - 1$. (3): We modify the proof of Proposition 3.7, where we have to replace the term $\Delta_f(\nu)\, p^\nu$ by $\Delta'_{f,\delta}(\nu)\, p^{\delta(\nu)}$ for $\nu \geq 3$. Since $\vartheta_{f,\delta} \geq 3$ and using (2), f satisfies (3.4) by the following arguments. We consider the inequality

$$ r = \delta(\nu) - 1 - \lfloor \log_p \nu \rfloor \geq 1, \quad \nu \geq 3, $$

which is satisfied by assumption. This gives the condition in (3.5), where we only need that $r \geq 1$. The congruences above follow similarly as in the proof of Proposition 3.7, which imply that $f \in \mathcal{K}_{p,1}$. ∎

This gives the notion to define the following classes of δ-degenerate functions.

Definition 5.3. *We define the sets*

$$ \mathcal{K}_{p,1}^{\mathrm{d}} = \{f \in \mathcal{C}(\mathbb{Z}_p) : f \text{ is } \delta\text{-degenerate}, \vartheta_{f,\delta} \geq 3, \delta(\nu) \geq 2 + \lfloor \log_p \nu \rfloor \text{ for } \nu \geq 3\}, $$
$$ \widehat{\mathcal{K}}_{p,1}^{\mathrm{d}} = \{f \in \mathcal{K}_{p,1}^{\mathrm{d}} : \Delta_f \neq 0, \text{additionally } 2 \mid \Delta_f(2) \text{ if } p = 2\}. $$

Corollary 5.4. *We have* $\mathcal{K}_{p,2} \subsetneq \mathcal{K}^{\mathrm{d}}_{p,1} \subset \mathcal{K}_{p,1}$ *and* $\widehat{\mathcal{K}}_{p,2} \subsetneq \widehat{\mathcal{K}}^{\mathrm{d}}_{p,1} \subset \widehat{\mathcal{K}}_{p,1}$.

Proof. Proposition 5.2 shows that $\mathcal{K}^{\mathrm{d}}_{p,1} \subset \mathcal{K}_{p,1}$. Let $f \in \widehat{\mathcal{K}}^{\mathrm{d}}_{p,1}$. Since $\Delta_f \neq 0$, we obtain $\left|(f(s)-f(t))/(p\,(s-t))\right|_p = 1$. Thus $f \in \widehat{\mathcal{K}}_{p,1}$. Define $\tilde{\delta}(\nu) = 2 + \lfloor \log_p \nu \rfloor$ for $\nu \geq 3$. Functions of $\mathcal{K}_{p,2}$ and $\widehat{\mathcal{K}}_{p,2}$ are also δ-degenerate with the strong property $\vartheta_{f,\delta} = \infty$ and $\delta(\nu) = \nu \geq \tilde{\delta}(\nu)$ for $\nu \geq 3$. But conversely, $f \in \widehat{\mathcal{K}}^{\mathrm{d}}_{p,1} \subset \mathcal{K}^{\mathrm{d}}_{p,1}$ with $\delta(\nu) = \tilde{\delta}(\nu)$ for $\nu \geq 3$ lies not in $\mathcal{K}_{p,2}$. ∎

Theorem 5.5. *If* $f \in \widehat{\mathcal{K}}^{\mathrm{d}}_{p,1}$ *and* $f(0) \in p\mathbb{Z}_p$, *then* f *has a unique simple zero* $\xi \in \mathbb{Z}_p$ *and*

$$f(s) = p\,(s-\xi)\, f^*(s), \quad |f(s)|_p = |p\,(s-\xi)|_p\,, \quad s \in \mathbb{Z}_p,$$

where $f^*(s) \equiv \Delta_f \pmod{p\mathbb{Z}_p}$ *and* $f^* \in \mathcal{C}(\mathbb{Z}_p)^*$.

Proof. Proposition 5.2 shows that f/p satisfies the conditions of Proposition 4.2. ∎

To compute any values $\pmod{p^n\mathbb{Z}_p}$ of a δ-degenerate function, we can use again a finite Mahler expansion.

Proposition 5.6. *Let* f *be a* δ*-degenerate function. For* $n \geq 1$ *and* $s \in \mathbb{Z}_p$, *we have*

$$f(s) \equiv \sum_{\nu=0}^{\eta_{f,\delta}(n)-1} \Delta'_{f,\delta}(\nu)\, p^{\delta(\nu)} \binom{s}{\nu} \pmod{p^n\mathbb{Z}_p}.$$

Proof. This is a consequence that $\delta(\nu) \geq n$ for $\nu \geq \eta_{f,\delta}(n)$. ∎

We can modify Algorithm 4.6 to compute a zero of functions $f \in \widehat{\mathcal{K}}^{\mathrm{d}}_{p,1}$, where $f(0) \in p\mathbb{Z}_p$, in the following way.

Algorithm 5.7. *Let* $f \in \widehat{\mathcal{K}}^{\mathrm{d}}_{p,1}$, $f(0) \in p\mathbb{Z}_p$, *and* ξ *be the unique zero of* f. *Let* $n \geq 1$ *be fixed. Initially set* $\xi_0 = 0$ *and* $\tilde{\delta} \equiv -\Delta_f^{-1} \pmod{p\mathbb{Z}_p}$. *For each step* $r = 1, \ldots, n$ *proceed as follows. Compute*

$$\gamma_{r-1} \equiv \sum_{\nu=0}^{\eta_{f,\delta}(r+1)-1} \Delta'_{f,\delta}(\nu)\, p^{\delta(\nu)-1} \binom{\xi_{r-1}}{\nu} \pmod{p^r\mathbb{Z}_p}. \tag{5.1}$$

Then $\gamma_{r-1} \in p^{r-1}\mathbb{Z}_p$. *Set* $s_{r-1} \equiv \gamma_{r-1}\tilde{\delta}/p^{r-1} \pmod{p\mathbb{Z}_p}$ *where* $0 \leq s_{r-1} < p$. *Set* $\xi_r = \xi_{r-1} + s_{r-1}\,p^{r-1}$ *and go to the next step while* $r < n$. *Finally,* $\xi_n \equiv \xi \pmod{p^n\mathbb{Z}_p}$.

Proof. The function f/p satisfies the conditions of Proposition 4.2, which we use to compute the zero of f/p. For each step (4.2) provides the next digit of the p-adic expansion of ξ; to compute the term $f_{r-1}(0)$ we make use of Proposition 5.6 modified for f/p. Note that $\vartheta_{f,\delta} \geq 3$ and $\Delta_f(0) = f(0) \in p\mathbb{Z}_p$, so (5.1) is valid. ∎

We have already seen in the proofs of Proposition 4.10 and Lemma 4.11 examples of functions, that slightly violate the conditions to be in $\mathcal{K}_{p,2}$. This can be described more precisely. First we need some lemmas.

Lemma 5.8. *If $f \in \mathcal{C}(\mathbb{Z}_p)$ has the Mahler expansion*

$$f(s) = \sum_{\nu \geq 0} a_\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p,$$

with coefficients $a_\nu \in \mathbb{Z}_p$ and $|a_\nu|_p \to 0$, then

$$f(s+t) = \sum_{\nu \geq 0} \binom{s}{\nu} \sum_{k \geq 0} \binom{t}{k} a_{\nu+k}, \quad s, t \in \mathbb{Z}_p.$$

Proof. Since $|a_\nu|_p \to 0$ and the sequence of the partial sums of the Mahler expansion of f is uniformly convergent to f, we can rearrange the series. Using Vandermonde's convolution identity again, we obtain

$$\begin{aligned} f(s+t) &= \sum_{\nu \geq 0} a_\nu \binom{s+t}{\nu} \\ &= \sum_{\nu \geq 0} a_\nu \sum_{j+k=\nu} \binom{s}{j} \binom{t}{k} \\ &= \sum_{\nu \geq 0} \binom{s}{\nu} \sum_{k \geq 0} \binom{t}{k} a_{\nu+k} \end{aligned}$$

after rearranging the sums. □

Lemma 5.9. *Let f be a δ-degenerate function. Define $\tilde{f}(s) = f(s+t)$ for $s \in \mathbb{Z}_p$, where $t \in \mathbb{Z}_p$ is a fixed translation. Then also $\tilde{f}$ is a δ-degenerate function.*

Proof. We have to show that $\tilde{f}$ has a Mahler series, that suffices Definition 5.1 regarding δ. We use Lemma 5.8 and set $a_\nu = \Delta'_{f,\delta}(\nu)\, p^{\delta(\nu)}$. Thus

$$\tilde{f}(s) = f(s+t) = \sum_{\nu \geq 0} \tilde{a}_\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p,$$

where

$$\tilde{a}_\nu = \sum_{k \geq 0} \binom{t}{k} \Delta'_{f,\delta}(\nu+k)\, p^{\delta(\nu+k)}.$$

Since δ is a monotonically increasing function, we achieve that $\tilde{a}_\nu \in p^{\delta(\nu)}\mathbb{Z}_p$. □

The lemma above shows the significance, that δ has to be a monotonically increasing function. Otherwise, the lemma does not work, since the coefficients $\tilde{a}_\nu$ take successional values $p^{\delta(\nu+k)}$ into account.

Proposition 5.10. *Let f be a δ-degenerate function. Define $\tilde{f}(s) = f(s+t)$ for $s \in \mathbb{Z}_p$, where $t \in \mathbb{Z}_p$ is a fixed translation. Then $\vartheta_{\tilde{f},\delta} = \vartheta_{f,\delta}$ and*

$$\Delta_{\tilde{f}}(\nu) \equiv \Delta_f(\nu) \pmod{p\mathbb{Z}_p}, \quad 0 \leq \nu \leq \vartheta_{f,\delta} - 2.$$

Proof. We use Proposition 5.2 and Lemma 5.9. Since $\tilde{f}$ is also a δ-degenerate function, we have $\vartheta_{f,\delta} = \vartheta_{\tilde{f},\delta}$. Assume that $\vartheta_{f,\delta} \geq 2$. Note that $\delta(\nu) = \nu$ for $0 \leq \nu \leq \vartheta_{f,\delta} - 1$ and $\delta(\vartheta_{f,\delta}) = \delta(\vartheta_{f,\delta} - 1)$, since we have a defect at index $\vartheta_{f,\delta}$ of the Mahler expansion of f. This transfers to $\tilde{f}$, such that

$$\tilde{f}(s) \equiv \sum_{\nu=0}^{\vartheta_{\tilde{f},\delta}-2} \Delta_{\tilde{f}}(\nu)\, p^\nu \binom{s}{\nu} \pmod{p^{\vartheta_{\tilde{f},\delta}-1}\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.$$

On the other side, we have for $0 \leq \nu \leq \vartheta_{f,\delta} - 2$ that

$$\Delta_{\tilde{f}}(\nu)\, p^\nu = \sum_{k \geq 0} \binom{t}{k} \Delta'_{f,\delta}(\nu + k)\, p^{\delta(\nu+k)} = \Delta_f(\nu)\, p^\nu + \mathcal{O}(p^{\nu+1}). \qquad \square$$

Corollary 5.11. *Let $f \in \mathcal{K}_{p,2}$. Define $\tilde{f}(s) = f(s+t)$ for $s \in \mathbb{Z}_p$, where $t \in \mathbb{Z}_p$ is a fixed translation. Then*

$$\Delta_{\tilde{f}}(\nu) \equiv \Delta_f(\nu) \pmod{p\mathbb{Z}_p}, \quad \nu \geq 0.$$

Proof. This follows by Proposition 5.10 and choosing $\delta = \mathrm{id}_{\mathbb{N}_0}$, so that $\vartheta_{f,\delta} = \infty$. $\square$

As a result the coefficients $\Delta_f(\nu)$ of functions $f \in \mathcal{K}_{p,2}$ are invariant $\pmod{p\mathbb{Z}_p}$ under translation. If f is a δ-degenerate function, then this property is valid up to index $\vartheta_{f,\delta} - 2$.

Proposition 5.12. *Assume that $p > 3$. If $f \in \mathcal{K}_{p,2}$ with $f(0) = 0$, then $f(s) = ps\, g(s)$ for $s \in \mathbb{Z}_p$, where $g \in \mathcal{K}_{p,1}^{\mathrm{d}}$.*

Proof. By Lemma 4.11 and (4.7) we have

$$g(s) = \sum_{\nu \geq 0} \frac{\Delta_f(\nu+1)\, p^\nu}{\nu + 1} \binom{s-1}{\nu}, \quad s \in \mathbb{Z}_p. \tag{5.2}$$

Lemma 5.9 shows that we can work with $\tilde{g}(s) = g(s+1)$ instead of g. Thus

$$\tilde{g}(s) = \sum_{\nu \geq 0} \Delta'_{\tilde{g},\delta}(\nu)\, p^{\delta(\nu)} \binom{s}{\nu}, \quad s \in \mathbb{Z}_p, \tag{5.3}$$

where

$$\Delta'_{\tilde{g},\delta}(\nu)\, p^{\delta(\nu)} = \Delta_f(\nu+1)\, p^\nu / (\nu+1).$$

One easily observes for $p > 3$ that $\vartheta_{\tilde{g},\delta} \geq p - 1 > 3$ and a simple counting argument shows that $\delta(\nu) \geq 2 + \lfloor \log_p \nu \rfloor$ for $\nu \geq 3$. Finally, $\tilde{g} \in \mathcal{K}_{p,1}^{\mathrm{d}}$ and equivalently $g \in \mathcal{K}_{p,1}^{\mathrm{d}}$. $\square$

Corollary 5.13. *Assume that $p > 3$. Let $f \in \mathcal{K}_{p,2}$ having a zero $\xi \in \mathbb{Z}_p$. Then $f(s) = p\,(s - \xi)\, g(s)$ for $s \in \mathbb{Z}_p$, where $g \in \mathcal{K}_{p,1}^{\mathrm{d}}$.*

Proof. Set $\tilde{f}(s) = f(s + \xi)$ and $\tilde{g}(s) = g(s + \xi)$. Then $\tilde{f}(s) = ps\, \tilde{g}(s)$. Note that $\tilde{f} \in \mathcal{K}_{p,2}$. This already follows by the definition of $\mathcal{K}_{p,2}$, due to the fact that $\nabla^n f(s) \equiv 0 \pmod{p^n\mathbb{Z}_p}$ for all $s \in \mathbb{Z}_p$ and $n \geq 1$. Applying Proposition 5.12, we get $\tilde{g} \in \mathcal{K}_{p,1}^{\mathrm{d}}$ and consequently $g \in \mathcal{K}_{p,1}^{\mathrm{d}}$ by Lemma 5.9. $\square$

Remark. For $p > 3$ and $f \in \mathcal{K}_{p,2}$, which has a zero $\xi \in \mathbb{Z}_p$, we have at least that $g(s) = f(s)/p(s-\xi)$ is a function of $\mathcal{K}_{p,1}^{\mathrm{d}}$. This shows that g obeys at least the Kummer congruences. Moreover, f can have two roots in $\mathbb{Z}_p$ under certain conditions as follows.

Definition 5.14. *We define the set*

$$\mathcal{K}_{p,2}^{2} = \left\{ f \in \mathcal{K}_{p,2} : \Delta_f = 0, \Delta_f(2) \in \mathbb{Z}_p^*, f \text{ has a zero in } \mathbb{Z}_p \right\}, \quad p > 3.$$

Theorem 5.15. *Assume that $p > 3$. If $f \in \mathcal{K}_{p,2}^{2}$, then f has two zeros $\xi_1, \xi_2 \in \mathbb{Z}_p$, such that*

$$f(s) = p^2\,(s-\xi_1)(s-\xi_2)\,f^*(s), \quad |f(s)|_p = \left|p^2\,(s-\xi_1)(s-\xi_2)\right|_p, \quad s \in \mathbb{Z}_p,$$

where $f^*(s) \equiv \Delta_f(2)/2 \pmod{p\mathbb{Z}_p}$ and $f^* \in \mathcal{C}(\mathbb{Z}_p)^*$.

Proof. By assumption f has a zero in $\mathbb{Z}_p$, say $\xi_1 \in \mathbb{Z}_p$. By Corollary 5.13 we then have the decomposition $f(s) = p\,(s-\xi_1)\,g(s)$, where $g \in \mathcal{K}_{p,1}^{\mathrm{d}}$. Again set $\tilde{f}(s) = f(s+\xi_1)$ and $\tilde{g}(s) = g(s+\xi_1)$. By Corollary 5.11 we have

$$\Delta_{\tilde{f}}(\nu) \equiv \Delta_f(\nu) \pmod{p\mathbb{Z}_p}, \quad \nu \geq 0.$$

According to (5.2) and (5.3) we obtain

$$\hat{g}(s) = \tilde{g}(s+1) = \sum_{\nu \geq 0} \frac{\Delta_{\tilde{f}}(\nu+1)\,p^\nu}{\nu+1} \binom{s}{\nu}, \quad s \in \mathbb{Z}_p.$$

As a result of Proposition 5.12, we know that $\hat{g} \in \mathcal{K}_{p,1}^{\mathrm{d}}$ and $\vartheta_{\hat{g},\delta} \geq 3$. Using Proposition 5.10, the above equation yields that

$$\begin{aligned} \Delta_g(0) &\equiv \Delta_{\hat{g}}(0) \equiv \Delta_{\tilde{f}}(1) \quad\ \equiv \Delta_f(1) \quad\ \equiv 0 \pmod{p\mathbb{Z}_p}, \\ \Delta_g(1) &\equiv \Delta_{\hat{g}}(1) \equiv \Delta_{\tilde{f}}(2)/2 \equiv \Delta_f(2)/2 \not\equiv 0 \pmod{p\mathbb{Z}_p}. \end{aligned}$$

Therefore g satisfies the conditions of Theorem 5.5, which provides

$$g(s) = p\,(s-\xi_2)\,g^*(s), \quad s \in \mathbb{Z}_p,$$

where $g^*(s) \equiv \Delta_g \pmod{p\mathbb{Z}_p}$ and $g^* \in \mathcal{C}(\mathbb{Z}_p)^*$. We finally set $f^* = g^*$ and observe that $\Delta_g \equiv \Delta_f(2)/2 \pmod{p\mathbb{Z}_p}$. This gives the result. $\square$

6. Ring properties and products

Theorem 6.1. *The p-adic Kummer spaces $\mathcal{K}_{p,1}$ and $\mathcal{K}_{p,2}$ are commutative rings, where $\mathcal{K}_{p,1}^*$ and $\mathcal{K}_{p,2}^*$, as defined in Definition 4.4, are their unit groups, respectively.*

Proof. Case $\mathcal{K}_{p,1}$: Multiplication: Let $f, g \in \mathcal{K}_{p,1}$ and $s, t \in \mathbb{Z}_p$. For $s \equiv t \pmod{p^n\mathbb{Z}_p}$ we get by Definition 3.1 that $f(s)g(s) \equiv f(t)g(t) \pmod{p^{n+1}\mathbb{Z}_p}$. Thus $f \cdot g \in \mathcal{K}_{p,1}$, where $(f \cdot g)(s) = f(s)g(s)$. Units: Since $f(0) \equiv f(s) \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$, we have $f^{-1}(s) \in \mathbb{Z}_p^*$ if and only if $f(0) \in \mathbb{Z}_p^*$. Let $f \in \mathcal{K}_{p,1}^*$, then we also have $f^{-1}(s) \equiv f^{-1}(t) \pmod{p^{n+1}\mathbb{Z}_p}$ when $s \equiv t \pmod{p^n\mathbb{Z}_p}$. Hence $f^{-1} \in \mathcal{K}_{p,1}^*$.

Case $\mathcal{K}_{p,2}$: Multiplication: Let $f, g \in \mathcal{K}_{p,2}$ and $w = f \cdot g$, where $w(s) = f(s)g(s)$ for $s \in \mathbb{Z}_p$. The product of the Mahler expansions of f and g yields

$$\begin{aligned} w(s) &= \sum_{\nu \geq 0} \Delta_f(\nu)\, p^\nu \binom{s}{\nu} \cdot \sum_{\nu \geq 0} \Delta_g(\nu)\, p^\nu \binom{s}{\nu} \\ &= \sum_{n \geq 0} p^n \sum_{n=j+k} \Delta_f(j)\, \Delta_g(k) \binom{s}{j}\binom{s}{k}. \end{aligned} \tag{6.1}$$

Now, for a fixed n, the polynomials $\binom{s}{j}\binom{s}{k}$ above have always degree n. By Lemma 2.6 we obtain that $\nabla^n w(s) \equiv 0 \pmod{p^n\mathbb{Z}_p}$. Since this is valid for all $n \geq 1$, we finally deduce that $w \in \mathcal{K}_{p,2}$. Units: Again $f(0) \equiv f(s) \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$ and $f^{-1}(s) \in \mathbb{Z}_p^*$ if and only if $f(0) \in \mathbb{Z}_p^*$. Let $f \in \mathcal{K}_{p,2}^*$. We have to show that $f^{-1} \in \mathcal{K}_{p,2}^*$ and necessarily that $\nabla^n f^{-1}(s) \equiv 0 \pmod{p^n\mathbb{Z}_p}$ for all $n \geq 1$. We will construct a sequence $(f_n)_{n\geq 1}$ of functions, such that $f_n \equiv f^{-1} \pmod{p^n\mathbb{Z}_p}$ and consequently $\lim_{n\to\infty} f_n = f^{-1}$. Let φ be Euler's totient function, then the Euler–Fermat's theorem reads

$$a^{\varphi(p^r)} \equiv 1 \pmod{p^r\mathbb{Z}_p} \quad \text{for } a \in \mathbb{Z}_p^*, r \geq 1. \tag{6.2}$$

Define $f_n = f^{\varphi(p^n)-1}$ for $n \geq 1$. Then $f_n \in \mathcal{K}_{p,2}^*$ and we have

$$\|f\|_p = \left\|f^{-1}\right\|_p = \|f_n\|_p = 1.$$

Using (6.2) we obtain for $n \geq r \geq 1$ and $s \in \mathbb{Z}_p$ that

$$f_n(s) \equiv f^{-1}(s) \pmod{p^r\mathbb{Z}_p}$$

and consequently

$$0 \equiv \nabla^r f_n(s) \equiv \nabla^r f^{-1}(s) \pmod{p^r\mathbb{Z}_p}.$$

Thus $\|f_n - f^{-1}\|_p \leq p^{-n}$ and $\|f_n - f^{-1}\|_p \to 0$ as $n \to \infty$. Finally $f^{-1} \in \mathcal{K}_{p,2}^*$.

All other ring axioms are also valid, since addition and multiplication are induced by standard operations. $\square$

Remark. Interestingly, the condition $2 \mid \Delta_f(2)$, as required in Definition 3.1 for functions $f \in \widehat{\mathcal{K}}_{p,2}$ in case $p = 2$, is preserved under ring operations among functions having this condition. We have to show that this is compatible with multiplication and inverse mapping, while it is trivial for addition. We use the same notation from above and adapt the proof. We show that the condition transfers to $\Delta_w(2)$ and $\Delta_{f^{-1}}(2)$, respectively. Multiplication: We have $\Delta_f(2) \equiv \Delta_g(2) \equiv 0 \pmod{2\mathbb{Z}_2}$ by assumption. Evaluating $\Delta_w(2)$ in (6.1) gives

$$\Delta_w(2) \equiv 2^{-2}\,\nabla^2 w(0) \equiv \Delta_f(0)\Delta_g(2) + 2\Delta_f(1)\Delta_g(1) + \Delta_f(2)\Delta_g(0) \equiv 0 \pmod{2\mathbb{Z}_2}.$$

Inverse mapping: The condition $2 \mid \Delta_f(2)$ is equivalent to $\nabla^2 f(0) \equiv 0 \pmod{2^3\mathbb{Z}_2}$. Since $f(s) \in \mathbb{Z}_2^*$, the values of f modulo 8 in question are in $\{1, 3, 5, 7\}$, which are inverse to themselves. Thus $\nabla^2 f^{-1}(0) \equiv \nabla^2 f(0) \equiv 0 \pmod{2^3\mathbb{Z}_2}$ and consequently $2 \mid \Delta_{f^{-1}}(2)$.

Remark. Since $\mathcal{K}_{p,2} \subset \mathcal{K}_{p,1}$, $\mathcal{K}_{p,2}^*$ is a subgroup of $\mathcal{K}_{p,1}^*$. Moreover $\mathbb{Z}_p^*$, identified as the group of constant functions, is a subgroup of $\mathcal{K}_{p,1}^*$ and $\mathcal{K}_{p,2}^*$.

Proposition 6.2. *Let* $\mathcal{U}_p = 1 + p\mathbb{Z}_p$. *The group of functions*
$$\mathcal{E}_p = \{f_a(s) = a^s : s \in \mathbb{Z}_p,\ a \in \mathcal{U}_p\}$$
is a subgroup of $\mathcal{K}_{p,2}^*$.

Proof. Obviously, we have $\mathcal{U}_p \cong \mathcal{E}_p$ as multiplicative groups. Following [26, Ch. 4.2, p. 173], the binomial expansion can be extended to a uniformly convergent Mahler series for $s \in \mathbb{Z}_p$:
$$f_a(s) = (a - 1 + 1)^s = \sum_{\nu \geq 0} (a-1)^\nu \binom{s}{\nu}.$$
Observing that $\nabla^\nu f_a(0) = (a-1)^\nu \in p^\nu \mathbb{Z}_p$, we can also write
$$f_a(s) = \sum_{\nu \geq 0} \Delta_{f_a}(\nu)\, p^\nu \binom{s}{\nu}.$$
Since $f_a(0) = 1$, it follows that $f_a \in \mathcal{K}_{p,2}^*$. $\square$

Corollary 6.3. *If* $a \in \mathcal{U}_p$, *then the equation*
$$a^\tau = \tau$$
is uniquely solvable with $\tau \in \mathcal{U}_p$, *which is effectively computable by Algorithm 4.8. Moreover, the map*
$$\kappa : \mathcal{U}_p \to \mathcal{U}_p, \quad a \mapsto \tau,$$
is injective.

Proof. Since the corresponding function $f_a \in \mathcal{K}_{p,2}$, Theorem 4.5 asserts that f_a has a fixed point. Thus, the equation above is uniquely solvable. From $1 = f_a(0) \equiv f_a(\tau) = \tau \pmod{p\mathbb{Z}_p}$, we deduce that $\tau \in \mathcal{U}_p$. Now we show that the map κ is injective. Let $a, b \in \mathcal{U}_p$, where $a \neq b$. Assume that $\kappa(a) = \kappa(b) = \tau$. Thus we get $(ab^{-1})^\tau = 1$, which implies that $a = b$ by the following Lemma 6.4. Contradiction. $\square$

Lemma 6.4. *Let* $a, \tau \in \mathcal{U}_p$, *then*
$$a^\tau = 1 \quad \Longleftrightarrow \quad a = 1.$$

Proof. If $a = 1$, then we are ready. Assume that $a = 1 + p^r \alpha$ with some $\alpha \in \mathbb{Z}_p^*$ and $r \geq 1$. Then
$$0 = a^\tau - 1 = \sum_{\nu \geq 1} p^{\nu r} \alpha^\nu \binom{\tau}{\nu} \equiv p^r \alpha \tau \pmod{p^{2r}\mathbb{Z}_p}.$$
Since $\tau \in \mathcal{U}_p$, we obtain $\alpha \equiv 0 \pmod{p^r \mathbb{Z}_p}$. Contradiction. $\square$

Using the ring properties of $\mathcal{K}_{p,2}$, we can give the following applications.

Proposition 6.5. *Let* $f \in \mathcal{K}_{p,2}$, *resp.*, $f \in \mathcal{K}_{p,2}^*$ *and* $r \in \mathbb{N}$, *resp.*, $r \in \mathbb{Z}$. *For* $s \in \mathbb{Z}_p$ *and* $n \geq 0$ *we have*
$$\left(\sum_{\nu=0}^{n} f(\nu) \binom{s}{\nu} \binom{n-s}{n-\nu} \right)^r \equiv \sum_{\nu=0}^{n} f(\nu)^r \binom{s}{\nu} \binom{n-s}{n-\nu} \pmod{p^{n+1}\mathbb{Z}_p}.$$

Proof. Since $\mathcal{K}_{p,2}$ is a ring, we have $f^r \in \mathcal{K}_{p,2}$, resp., $f^r \in \mathcal{K}^*_{p,2}$. By Proposition 3.10 both sides of the congruence above are congruent to $f(s)^r$. $\square$

Proposition 6.6. *Let $f, g \in \mathcal{K}_{p,2}$, $s \in \mathbb{Z}_p$, and $n \geq 1$. Define the convolution*

$$\nabla^n (f \star g)(s) = \sum_{\nu=0}^{n} \binom{n}{\nu} (-1)^{n-\nu} f(s+\nu) g(s+n-\nu).$$

Then $\nabla^n (f \star g)(s) \equiv 0 \pmod{p^n \mathbb{Z}_p}$.

Proof. Define $\hat{g} = g \circ \lambda$, where $\lambda(s) = 2\hat{s} + n - s$ with a fixed $\hat{s} \in \mathbb{Z}_p$. Then $\hat{g}(s+\nu) = g(2\hat{s} + n - s - \nu)$. By Lemma 3.13 we know that $\hat{g} \in \mathcal{K}_{p,2}$. Thus $f \cdot \hat{g} \in \mathcal{K}_{p,2}$. Now, let s be fixed and choose $\hat{s} = s$. We finally obtain in this case that

$$\nabla^n (f \star g)(s) = \nabla^n (f \cdot \hat{g})(s) \equiv 0 \pmod{p^n \mathbb{Z}_p}. \qquad \square$$

Now, we shall examine properties of products of functions of $\mathcal{K}_{p,2}$.

Proposition 6.7. *Let $n \geq 1$ and*

$$F = \prod_{\nu=1}^{n} f_\nu, \quad f_\nu \in \mathcal{K}_{p,2}.$$

Then

$$\Delta_F(0) = \prod_{\nu=1}^{n} \Delta_{f_\nu}(0)$$

and

$$\Delta_F(k) \equiv \sum_{\nu_1 + \cdots + \nu_n = k} \binom{k}{\nu_1, \ldots, \nu_n} \Delta_{f_1}(\nu_1) \cdots \Delta_{f_n}(\nu_n) \pmod{p \mathbb{Z}_p}$$

for $k \geq 1$.

Proof. In view of (6.1), we obtain more generally that

$$F(s) = \sum_{k \geq 0} p^k \sum_{\nu_1 + \cdots + \nu_n = k} \Delta_{f_1}(\nu_1) \cdots \Delta_{f_n}(\nu_n) \binom{s}{\nu_1} \cdots \binom{s}{\nu_n}.$$

Since $F \in \mathcal{K}_{p,2}$, we have the Mahler expansion

$$F(s) = \sum_{k \geq 0} \Delta_F(k)\, p^k \binom{s}{k}.$$

For $k = 0$ we have $\Delta_F(0) = F(0) = f_1(0) \cdots f_n(0) = \Delta_{f_1}(0) \cdots \Delta_{f_n}(0)$. Let $k \geq 1$ be fixed, then $\pi(s) = (s)_{\nu_1} \cdots (s)_{\nu_n}$ is a monic polynomial of degree k. Note that $\nabla (s)_m = m\, (s)_{m-1}$

and $\nabla^k \pi(0) = k!$ by Lemma 2.6, but in general $\nabla^r \pi(0) \neq 0$ for $k > r \geq 1$, because $\nabla^r \pi(s)$ can have a constant term. Evaluating $\nabla^k F(0)/p^k$ we obtain

$$\Delta_F(k) = \sum_{\nu_1+\cdots+\nu_n=k} \binom{k}{\nu_1,\ldots,\nu_n} \Delta_{f_1}(\nu_1)\cdots\Delta_{f_n}(\nu_n) + \sum_{k'>k} p^{k'-k} \sum_{\nu_1+\cdots+\nu_n=k'} \Delta_{f_1}(\nu_1)\cdots\Delta_{f_n}(\nu_n)\, \nabla^k \binom{s}{\nu_1}\cdots\binom{s}{\nu_n} \Big|_{s=0},$$

where the second sum converges on $\mathbb{Z}_p$ and vanishes $\pmod{p\mathbb{Z}_p}$. □

Definition 6.8. *We define for functions $f \in \mathcal{K}_{p,2}$ the following parameters*

$$\lambda_f = \min_{\nu \geq 0}\{\nu : \Delta_f(\nu) \in \mathbb{Z}_p^*\}, \qquad \mu_f = \max_{\nu \geq 0}\{\nu : f/p^\nu \in \mathcal{K}_{p,2}\}$$

and we set $\lambda_f = \mu_f = \infty$ in case $f = 0$. We further define

$$\mathcal{K}'_{p,2} = \{f \in \mathcal{K}_{p,2} : \lambda_f < \infty\}, \quad \mathcal{K}''_{p,2} = \{f/p^{\mu_f} : f \in \mathcal{K}_{p,2}\}.$$

Lemma 6.9. *We have $\mathcal{K}'_{p,2} = \mathcal{K}''_{p,2}$ and the decomposition $\mathcal{K}_{p,2} = \mathcal{K}'_{p,2} \times p^{\mathbb{N}_0}$.*

Proof. We have $\mathbb{Z}_p \subset \mathcal{K}_{p,2}$, since constant functions are in $\mathcal{K}_{p,2}$. Note that $1 \in \mathcal{K}'_{p,2}$ and $0, 1 \in p^{\mathbb{N}_0}$. Let $f \in \mathcal{K}_{p,2}$ where $f \neq 0$. In case $\lambda_f = \infty$ we can split the prime p from f to get $f/p \cdot p$. Then $\Delta_{f/p}(\nu) = \Delta_f(\nu)/p$ for all $\nu \geq 1$ and $f/p \in \mathcal{K}_{p,2}$. This procedure can be finitely repeated, say r times, until $f/p^r \in \mathcal{K}'_{p,2}$ and we are ready. In case $\lambda_f < \infty$ set $r = 0$. By construction r is maximal, so we get $r = \mu_f$ and $f/p^{\mu_f} \in \mathcal{K}'_{p,2}$. This shows the decomposition $\mathcal{K}_{p,2} = \mathcal{K}'_{p,2} \times p^{\mathbb{N}_0}$. Now, by the same arguments we achieve that $\mathcal{K}'_{p,2} = \mathcal{K}''_{p,2}$, since $r = \mu_f$ is maximal. The case $f = 0$ yields the term $0/0$, which we define to be 1. □

Proposition 6.10. *Let $n \geq 1$ and*

$$F = \prod_{\nu=1}^n f_\nu, \quad f_\nu \in \mathcal{K}'_{p,2}.$$

Then

$$\Delta_F(m) \equiv \binom{m}{\lambda_{f_1},\ldots,\lambda_{f_n}} \prod_{\nu=1}^n \Delta_{f_\nu}(\lambda_{f_\nu}) \pmod{p\mathbb{Z}_p},$$

where

$$\lambda_F \geq m = \sum_{\nu=1}^n \lambda_{f_\nu}.$$

Moreover, $\lambda_F > m$ if and only if

$$\binom{m}{\lambda_{f_1},\ldots,\lambda_{f_n}} \in p\mathbb{Z}_p.$$

Proof. From Proposition 6.7 we have for $k \geq 0$ that

$$\Delta_F(k) \equiv \sum_{\nu_1+\cdots+\nu_n=k} \binom{k}{\nu_1,\ldots,\nu_n} \Delta_{f_1}(\nu_1) \cdots \Delta_{f_n}(\nu_n) \pmod{p\mathbb{Z}_p}.$$

Case $k < m$: Since $\Delta_{f_j}(\nu) \in p\mathbb{Z}_p$ for $\lambda_{f_j} > \nu \geq 0$ and $\Delta_{f_j}(\lambda_{f_j}) \in \mathbb{Z}_p^*$ for $j \in \{1,\ldots,n\}$, we observe that $\Delta_F(k) \in p\mathbb{Z}_p$ for $m > k \geq 0$. Case $k = m$: All terms of the sum, except for the term where $\nu_1 = \lambda_{f_1}, \ldots, \nu_n = \lambda_{f_n}$, vanish $\pmod{p\mathbb{Z}_p}$. This gives the proposed formula for $\Delta_F(m)$. Let $b = \binom{m}{\lambda_{f_1},\ldots,\lambda_{f_n}}$. If $b \in \mathbb{Z}_p^*$, then $\lambda_F = m$. Otherwise $b \in p\mathbb{Z}_p$ implies that $\lambda_F > m$. $\square$

Corollary 6.11. *Let $f \in \mathcal{K}'_{p,2}$, $u \in \mathcal{K}^*_{p,2}$, and $g = fu$. The parameter λ_f is invariant under multiplication of f and units u:*

$$\lambda_g = \lambda_f, \quad \Delta_g(\lambda_g) \equiv \Delta_f(\lambda_f)\, u(0) \pmod{p\mathbb{Z}_p}.$$

Proof. Using Proposition 6.10, we get $m = \lambda_f + \lambda_u = \lambda_f$ and

$$\Delta_g(m) \equiv \binom{m}{m} \Delta_f(\lambda_f) \Delta_u(\lambda_u) \equiv \Delta_f(\lambda_f)\, u(0) \not\equiv 0 \pmod{p\mathbb{Z}_p}.$$

Thus $\lambda_g = m = \lambda_f$. $\square$

Theorem 6.12. *Let $n \geq 1$ and*

$$F = \prod_{\nu=1}^{n} f_\nu, \quad f_\nu \in \widehat{\mathcal{K}}^0_{p,2}.$$

Then

$$F(s) = p^n \prod_{\nu=1}^{n} (s - \xi_\nu) \cdot F^*(s), \quad s \in \mathbb{Z}_p,$$

where $\xi_\nu \in \mathbb{Z}_p$ is the zero of f_ν. Moreover,

$$F^*(s) \equiv \prod_{\nu=1}^{n} \Delta_{f_\nu} \pmod{p\mathbb{Z}_p}, \quad F^* \in \mathcal{C}(\mathbb{Z}_p)^*, \quad s \in \mathbb{Z}_p,$$

and

$$\Delta_F(n) \equiv n! \prod_{\nu=1}^{n} \Delta_{f_\nu} \pmod{p\mathbb{Z}_p}.$$

Moreover, $\operatorname{ord}_p \Delta_F(\nu) \geq n - \nu$ *for $\nu = 0, \ldots, n$. If $n < p$, then $\lambda_F = n$, otherwise $\lambda_F > n$.*

Proof. By Theorem 4.5 we have the following representation for a function $f_\nu \in \widehat{\mathcal{K}}^0_{p,2}$: $f_\nu(s) = p\,(s - \xi_\nu)\, f^*_\nu(s)$, where $f^*_\nu(s) \equiv \Delta_{f_\nu} \not\equiv 0 \pmod{p\mathbb{Z}_p}$. Thus the product representation of F and F^* follows easily. Since $\lambda_{f_\nu} = 1$ we get from Proposition 6.10 a simplified formula $\Delta_F(n) \equiv n! \prod_{\nu=1}^{n} \Delta_{f_\nu} \pmod{p\mathbb{Z}_p}$. Hence $\lambda_F = n$ if $n < p$. For $n \geq p$ we obtain $\Delta_F(n) \equiv 0 \pmod{p\mathbb{Z}_p}$, which implies that $\lambda_F > n$ in that case. Now, $\operatorname{ord}_p \Delta_F(\nu) \geq n - \nu$ for $\nu = 0, \ldots, n$ follows by $\Delta_F(\nu) = \nabla^\nu F(0)/p^\nu = p^{n-\nu}\, \nabla^\nu \tilde{F}(0)$, where $\tilde{F} = F/p^n$ is a function on $\mathbb{Z}_p$ as seen above. $\square$

Definition 6.13. *We define the class of monic polynomial like functions by*

$$\mathcal{K}_{p,2}^{\mathrm{s}} = \left\{ f \in \mathcal{K}_{p,2} : f = \prod_{\nu=1}^{n} f_\nu,\ n \geq 1,\ f_\nu \in \widehat{\mathcal{K}}_{p,2}^{0} \right\}$$

having all roots in $\mathbb{Z}_p$.

We get an analogue to the p-adic Weierstrass Preparation Theorem.

Corollary 6.14. *If $f \in \mathcal{K}_{p,2}^{\mathrm{s}}$ and $\lambda_f < p$, then we have the decomposition*

$$f = p^{\lambda_f} \times \pi_f \times u,$$

where π_f is a monic polynomial of degree λ_f, that splits over $\mathbb{Z}_p$, and $u \in \mathcal{C}(\mathbb{Z}_p)^*$.

Proof. This follows from Theorem 6.12, since $\lambda_f = n < p$. $\square$

Again, we have a close relation between the zeros and the fixed point of $f \in \mathcal{K}_{p,2}^{\mathrm{s}}$, which extends Lemma 4.9.

Corollary 6.15. *If $f \in \mathcal{K}_{p,2}^{\mathrm{s}}$ where $f(0) \neq 0$ and $n = \lambda_f < p$, then*

$$(-1)^n \frac{\Delta_f(n)}{n!} \equiv \frac{\tau}{p^n \prod_{\nu=1}^{n} \xi_\nu} \pmod{p\mathbb{Z}_p}, \quad f(0)/\tau \equiv 1 \pmod{p\mathbb{Z}_p},$$

and

$$\operatorname{ord}_p f(0) = \operatorname{ord}_p \tau = n + \sum_{\nu=1}^{n} \operatorname{ord}_p \xi_\nu,$$

where τ is the fixed point and ξ_ν are the zeros of f.

Proof. We have excluded the case $f(0) = 0$ which implies $\tau = 0$ and vice versa, so $\tau \neq 0$ and $f(0) \neq 0$. Since $n < p$ we obtain by Theorem 6.12 that

$$f(0) = (-1)^n f^*(0)\, p^n \prod_{\nu=1}^{n} \xi_\nu, \quad \tau = f^*(\tau)\, p^n \prod_{\nu=1}^{n} (\tau - \xi_\nu),$$

where $f^*(s) \equiv \Delta_f(n)/n! \not\equiv 0 \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$. Thus

$$f^*(\tau)^{-1} \equiv \frac{p^n \prod_{\nu=1}^{n} (\tau - \xi_\nu)}{\tau} \equiv (-1)^n p^n \prod_{\nu=1}^{n} \xi_\nu \Big/ \tau \pmod{p\mathbb{Z}_p},$$

where we have used the following argument. Expanding the product above, we get summands s_j which have a factor τ. For those terms we see that $p^n s_j / \tau$ vanishes $(\mathrm{mod}\ p\mathbb{Z}_p)$. Consequently, it only remains the product over the zeros as given above. The rest follows easily. $\square$

Functions of $\mathcal{K}_{p,2}^{\mathrm{s}}$ have a controlled but unbounded growth when viewed in the p-adic norm via $|\cdot|_p^{-1}$, since all roots lie in $\mathbb{Z}_p$. We can also consider arbitrary monic polynomials, where the roots may lie in some finite extension of $\mathbb{Q}_p$.

Definition 6.16. *We define the class of monic polynomial like functions by*

$$\begin{aligned}\mathcal{K}_{p,2}^{\mathrm{m}} = \{ f \in \mathcal{K}_{p,2} : f = p^n\, \pi_n\, u, \text{ monic } \pi_n \in \mathbb{Z}_p[s], \\ \deg \pi_n = n \geq 1,\ u \in \mathcal{K}_{p,2}^* \}.\end{aligned}$$

Proposition 6.17. *If $f \in \mathcal{K}_{p,2}^{\mathrm{m}}$ where $f = p^n\, \pi_n\, u$, then $\lambda_f = n$ in case $n < p$, otherwise $\lambda_f > n$. Moreover, if $n \geq 2$ and π_n is irreducible over $\mathbb{Z}_p[s]$, then there exists a lower bound*

$$|f(s)|_p \geq p^{-c}, \quad s \in \mathbb{Z}_p,$$

with some constant c where $n \leq c < \infty$ depending on f.

Proof. First we look at $\tilde{f} = p^n\, \pi_n$. Since we have $\nabla^n \pi_n(0) = n!$, we deduce that $\lambda_{\tilde{f}} = n$ in case $n < p$, otherwise $\lambda_{\tilde{f}} > n$. This property transfers to λ_f by Corollary 6.11. Now, let $n \geq 2$ and π_n be irreducible over $\mathbb{Z}_p[s]$. We have $|f(s)|_p = p^{-n}\, |\pi_n(s)|_p$. Assume that $\min_{s \in \mathbb{Z}_p} |f(s)|_p$ is unbounded. Since $\mathbb{Z}_p$ is compact and has a discrete valuation, we would get $|f(s')|_p = 0$ for some $s' \in \mathbb{Z}_p$. This implies that $\pi_n(s') = 0$ and s' is a root of π_n in $\mathbb{Z}_p$. This gives a contradiction. ∎

Remark. Products of functions $f(s) = p\,(s - \xi)\, u(s)$, where $\xi \in \mathbb{Z}_p$ and $u \in \mathcal{K}_{p,2}^*$, are in $\mathcal{K}_{p,2}^{\mathrm{s}} \cap \mathcal{K}_{p,2}^{\mathrm{m}}$. But we have $\mathcal{K}_{p,2}^{\mathrm{s}} \not\subset \mathcal{K}_{p,2}^{\mathrm{m}}$ as a consequence of Proposition 4.10.

At last, we construct a class of functions that are constant regarding the p-adic norm.

Definition 6.18. *We define the class of constant functions regarding the p-adic norm by*

$$\begin{aligned}\mathcal{K}_{p,2}^{\mathrm{c}} = \{ f \in \mathcal{K}_{p,2} : \operatorname{ord}_p f(0) = n < \lambda_f, \\ \operatorname{ord}_p \Delta_f(\nu) > n - \nu \text{ for } \nu = 1, \ldots, n \}.\end{aligned}$$

Proposition 6.19. *If $f \in \mathcal{K}_{p,2}^{\mathrm{c}}$, then $|f(s)|_p = p^{-n}$ for $s \in \mathbb{Z}_p$, where $n = \operatorname{ord}_p f(0) > 0$.*

Proof. Let $\operatorname{ord}_p f(0) = n$. Since $\lambda_f > n$ by definition, the case $n = 0$ is not possible, so $n \geq 1$. Set $\Delta'_f(\nu) = \Delta_f(\nu)/p^{n+1-\nu} \in \mathbb{Z}_p$ for $\nu = 1, \ldots, n$. Then we get

$$f(s) = f(0) + \sum_{\nu=1}^{n} \Delta'_f(\nu)\, p^{n+1} \binom{s}{\nu} + \sum_{\nu > n} \Delta_f(\nu)\, p^{\nu} \binom{s}{\nu}.$$

Thus $f(s) = f(0) + \mathcal{O}(p^{n+1})$. Since $\operatorname{ord}_p f(0) = n$, we have $p^{n+1} \nmid f(s)$ and consequently $|f(s)|_p = p^{-n}$. ∎

Table 6.20. Classification of $\mathcal{K}_{p,2}$.

$f \in$	λ_f	$\operatorname{ord}_p f(0)$	$\lvert f(s)\rvert_p$
$\mathcal{K}_{p,2}^*$	0	0	1
$\widehat{\mathcal{K}}_{p,2}^0$	1	≥ 1	$\lvert p\,(s-\xi)\rvert_p$, $\xi \in \mathbb{Z}_p$
$\mathcal{K}_{p,2}^2$	2	≥ 2	$\lvert p^2\,(s-\xi_1)(s-\xi_2)\rvert_p$, $\xi_1, \xi_2 \in \mathbb{Z}_p$
$\mathcal{K}_{p,2}^{\mathrm{s}}$	$\lambda_f \geq n$	$\geq n$	$\prod_{\nu=1}^n \lvert p\,(s-\xi_\nu)\rvert_p$, $\xi_\nu \in \mathbb{Z}_p$
$\mathcal{K}_{p,2}^{\mathrm{m}}$	$\lambda_f \geq n$	$\geq n$	$\lvert p^n\, \pi_n(s)\rvert_p$, $\deg \pi_n = n$
$\mathcal{K}_{p,2}^{\mathrm{c}}$	$\lambda_f > n > 0$	n	p^{-n}

Functions $f \in \mathcal{K}_{p,2}$, that have the property $\operatorname{ord}_p f(0) = 1$, can be described as follows. As a result, such functions, not being in $\widehat{\mathcal{K}}^0_{p,2}$, are constant regarding the p-adic norm. We exclude the case $p = 2$, since the additional condition $2 \mid \Delta_f(2)$ makes some difficulties.

Proposition 6.21. *Let $p \geq 3$. If $f \in \mathcal{K}_{p,2}$ with $\operatorname{ord}_p f(0) = 1$, then f has exactly one of the following forms:*

$f \in$	μ_f	λ_f	$\lvert f(s)\rvert_p,\ s \in \mathbb{Z}_p$
$\widehat{\mathcal{K}}^0_{p,2}$	0	1	$\lvert p\,(s-\xi)\rvert_p,\ \xi \in \mathbb{Z}_p^*$
$\mathcal{K}^{\mathrm{c}}_{p,2}$	0	≥ 2	p^{-1}
$p\mathcal{K}^*_{p,2}$	1	∞	p^{-1}

Proof. First, if we have $\mu_f > 0$, then only $f \in p\mathcal{K}^*_{p,2}$ is possible, since $\operatorname{ord}_p f(0) = 1$. Now, we can assume that $\mu_f = 0$ and $1 \leq \lambda_f < \infty$. The Mahler expansion shows that

$$f(s) \equiv f(0) + \Delta_f(1)\, ps \pmod{p^2\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.$$

We have the cases $\Delta_f(1) \in \mathbb{Z}_p^*$ and $\Delta_f(1) \in p\mathbb{Z}_p$. The first case implies that $\lambda_f = 1$ and $f \in \widehat{\mathcal{K}}^0_{p,2}$. By Theorem 4.5 we then get $\lvert f(s)\rvert_p = \lvert p\,(s-\xi)\rvert_p$ with some $\xi \in \mathbb{Z}_p$. Since $\lvert f(0)\rvert_p = \lvert p\,\xi\rvert_p = p^{-1}$, it even follows that $\xi \in \mathbb{Z}_p^*$. The second case provides that $\lambda_f \geq 2$ and f suffices the conditions of Definition 6.18 and Proposition 6.19 with $n = 1$. $\square$

We shall also show the more complicated case $\operatorname{ord}_p f(0) = 2$ of functions $f \in \mathcal{K}_{p,2}$. Since we have a decomposition of $\mathcal{K}_{p,2}$ by Lemma 6.9, we only consider those cases where $\mu_f = 0$ to simplify the results.

Proposition 6.22. *Let $p \geq 3$. If $f \in \mathcal{K}'_{p,2}$ with $\operatorname{ord}_p f(0) = 2$, then f has one of the following forms*

$f \in$	λ_f	$\lvert f(s)\rvert_p,\ s \in \mathbb{Z}_p$
$\widehat{\mathcal{K}}^0_{p,2}$	1	$\lvert p\,(s-\xi)\rvert_p,\ \xi \in \mathbb{Z}_p,\ \operatorname{ord}_p \xi = 1$
$\mathcal{K}^2_{p,2}$, $\mathcal{K}^{\mathrm{s}}_{p,2}$	2	$\lvert p^2\,(s-\xi_1)(s-\xi_2)\rvert_p,\ \xi_1, \xi_2 \in \mathbb{Z}_p^*$, $p > 3$ *if* $f \in \mathcal{K}^2_{p,2}$
$\mathcal{K}^{\mathrm{m}}_{p,2}$	2	$\lvert p^2\,\pi_2(s)\rvert_p \geq p^{-c},\ \pi_2(0) \in \mathbb{Z}_p^*$, π_2 *irreducible*, $c \geq 2$.
$\mathcal{K}^{\mathrm{c}}_{p,2}$	≥ 3	p^{-2}

or f has the behavior that $\lvert f(s)\rvert_p \leq p^{-2}$ for $s \in \mathbb{Z}_p$.

Proof. The Mahler expansion provides that

$$f(s) \equiv f(0) + \Delta_f(1)\, ps + \Delta_f(2)\, p^2 \binom{s}{2} \pmod{p^3\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.$$

Again, the case $\lambda_f = 1$ yields $\lvert f(s)\rvert_p = \lvert p\,(s-\xi)\rvert_p$ with $\operatorname{ord}_p \xi = 1$, caused by $\lvert f(0)\rvert_p = \lvert p\,\xi\rvert_p = p^{-2}$. The case $\lambda_f \geq 2$ implies that $\Delta_f(1) \in p\mathbb{Z}_p$ and we know at least that $\lvert f(s)\rvert_p \leq p^{-2}$ for $s \in \mathbb{Z}_p$. Due to $\operatorname{ord}_p f(0) = 2$, we obtain the supplementary classification as above by Theorem 5.15, Corollary 6.14, and Propositions 6.17 and 6.19. $\square$

7. p-adic interpolation of L-functions

Definition 7.1. *Let the function $f : \mathbb{N}_0 \to \mathbb{Z}_p$ satisfy the Kummer type congruences*

$$\nabla^n f(0) \equiv 0 \pmod{p^n \mathbb{Z}_p} \quad \text{for all } n \geq 0.$$

Then we call f a Kummer function.

The standard way to extend a function, which is defined for nonnegative integer arguments, to the domain $\mathbb{Z}_p$ is the following, cf. [19, Ch. 2]. A function $f : \mathbb{N}_0 \to \mathbb{Z}_p$ can be uniquely extended to a function $\tilde{f} : \mathbb{Z}_p \to \mathbb{Z}_p$, which interpolates values $\tilde{f}(n) = f(n)$ for nonnegative integers n. Then define for $s \in \mathbb{Z}_p$ that $\tilde{f}(s) = \lim_{t_\nu \to s} f(t_\nu)$ for any sequence $(t_\nu)_{\nu \geq 1}$ of nonnegative integers which p-adically converges to s. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, there exists at most one function $\tilde{f}$ with these properties. Finally, we can identify $f = \tilde{f}$. Note that, for example, $f(-1) = \lim_{n \to \infty} f(p^n - 1)$. If the function f satisfies the Kummer congruences, then f is continuous on $\mathbb{Z}_p$ and $f \in \mathcal{K}_{p,1}$.

Proposition 7.2. *Let f be a Kummer function. Then f can be uniquely extended to a continuous p-adic function on $\mathbb{Z}_p$ such that $f \in \mathcal{K}_{p,2}$.*

Proof. According to Definition 3.1, define the Mahler expansion

$$\tilde{f}(s) = \sum_{\nu \geq 0} \Delta_f(\nu)\, p^\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p.$$

By construction $\tilde{f} = f$ restricted on $\mathbb{N}_0$. Since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$, the same arguments from above are valid here. We get a p-adic function $\tilde{f}$ on $\mathbb{Z}_p$, such that the Mahler expansions of $\tilde{f}$ and f are equal and $\tilde{f}$ extends f uniquely to $\mathbb{Z}_p$. Lemma 3.2 shows that $\tilde{f} \in \mathcal{K}_{p,2}$. Finally, we identify $f = \tilde{f}$ as a function of $\mathcal{K}_{p,2}$. □

Remark. It should be noted that Sun [28] introduced so-called p-regular functions which are Kummer functions in this context. He proved some special congruences, which easily follow here in general and have a full interpretation in $\mathcal{K}_{p,2}$. However, his proofs are completely different, complicated, and lengthy using properties of Stirling numbers and the binomial inversion theorem.

Definition 7.3. *The generalized Bernoulli numbers $B_{n,\chi}$ are defined by the generating function*

$$\sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) \frac{z e^{az}}{e^{\mathfrak{f}_\chi z} - 1} = \sum_{n \geq 0} B_{n,\chi} \frac{z^n}{n!}, \quad |z| < \frac{2\pi}{\mathfrak{f}_\chi},$$

where χ is a primitive Dirichlet character $\pmod{\mathfrak{f}_\chi}$ *and $\mathfrak{f}_\chi$ is a positive integer, which is called the conductor of χ. Choose $\delta_\chi \in \{0, 1\}$ such that $\chi(-1) = (-1)^{\delta_\chi}$ and δ_χ corresponds to even, resp., odd characters. The Dirichlet L-functions are defined by*

$$L(z, \chi) = \sum_{\nu \geq 1} \chi(\nu) \nu^{-z}, \quad z \in \mathbb{C},\ \operatorname{Re} z > 1.$$

The numbers $B_{n,\chi}$ were introduced and studied by Leopoldt [22] and subsequently examined by Carlitz [4] and others. They fulfill the following basic properties, cf. [29, Ch. 4], where we have to exclude the case $n = 1$ when $\chi = 1$ since $B_{1,1} = -B_1 = \frac{1}{2}$:

$$\begin{aligned} \mathfrak{f}_\chi B_{n,\chi} &\in \mathbb{Z}[\chi], && \chi \neq 1, \\ B_{n,\chi} &\neq 0, && n \geq 1,\ n \equiv \delta_\chi \pmod 2, \\ B_{n,\chi} &= 0, && n \geq 1,\ n \not\equiv \delta_\chi \pmod 2, \\ L(1-n,\chi) &= -\frac{B_{n,\chi}}{n}, && n \geq 1. \end{aligned}$$

The values of $B_{n,\chi}$ are given by the Bernoulli polynomials $B_n(\cdot)$:

$$\begin{aligned} B_{1,\chi} &= \frac{1}{\mathfrak{f}_\chi} \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) a, && \chi \neq 1, \\ B_{n,\chi} &= \mathfrak{f}_\chi^{n-1} \sum_{a=1}^{\mathfrak{f}_\chi} \chi(a) B_n\left(\frac{a}{\mathfrak{f}_\chi}\right), && n > 1. \end{aligned}$$

Note that $B_{n,\chi}$ reduces to the Bernoulli numbers B_n for $n > 1$, if $\chi = 1$ is the principal character with conductor 1, where $\zeta(z) = L(z,1)$ is the Riemann zeta function. If $\chi = \chi_{-4}$ is the non-principal character $\pmod 4$, then $-2B_{n+1,\chi}/(n+1)$ reduces to the Euler numbers E_n.

Kubota and Leopoldt [20] constructed p-adic L-functions, that interpolate values of L-functions, modified by an Euler factor, at negative integer arguments. Here we regard their first construction of p-adic L-functions, that are defined on certain residue classes, cf. Koblitz [19]. Note that their second construction is connected with Iwasawa theory.

Recall Euler's totient function φ and set $q = p$ for $p \geq 3$ and $q = 4$ for $p = 2$, which we use in the following.

Definition 7.4. *Define the modified L-functions by*

$$L_p(1-n,\chi) = (1 - \chi(p)p^{n-1}) L(1-n,\chi), \quad n \geq 1.$$

We further define the modified L-functions on residue classes $\pmod{\varphi(q)}$ *by*

$$L_{p,l}(s,\chi) = L_p(1 - (\delta_\chi + l + \varphi(q)s), \chi), \quad s \in \mathbb{N}_0,$$

where l is a fixed integer and $0 \leq l \leq \varphi(q) - 2$. If $l = \delta_\chi = 0$, then we exclude the case $s = 0$. We write $\zeta_{p,l}(s)$ for $L_{p,l}(s,1)$. Define the backward variable substitution

$$s_{p,l}(n) = (n-l)/\varphi(q),$$

where we briefly write $s_{p,l}$ in case of no ambiguity.

Note that $L_{p,l}(\cdot,\chi)$ is defined regardless of the parity of the character χ, such that $L_{p,l}(\cdot,\chi)$ is the zero function for odd l. The generalized Bernoulli numbers $B_{n,\chi}/n$, resp., the L-functions $L_p(\cdot,\chi)$ at negative integer arguments satisfy the Kummer type congruences. As usual we have to omit the prime p where $\mathfrak{f}_\chi = p^e$ is a prime power with $e \geq 1$.

Theorem 7.5 (Carlitz [4], Fresnel [11])**.** *Let $\chi = 1$ or χ be a primitive non-principal character* $(\mathrm{mod}\ \mathfrak{f}_\chi)$*. Assume that $p^e \neq \mathfrak{f}_\chi$, $e \geq 1$. Let k, n, r be positive integers and $h = k\varphi(p^r)$ be even. Then*

$$\nabla_h^n L_p(1-s,\chi)\,\Big|_{s=m} \equiv 0 \pmod{p^{nr}}, \quad m \in \delta_\chi + 2\mathbb{N}_0,\ m \geq 1,$$

where in case $\chi = 1$ additionally suppose that $p > 3$ and $m \not\equiv 0 \pmod{p-1}$.

It should be noted that Carlitz rarely used Euler factors, so his congruences are restricted to $(\mathrm{mod}\ (p^{nr}, p^{m-1}))$ here. Generally, since $L_p(1-m,\chi) \in \mathbb{Q}(\chi) \subset \overline{\mathbb{Q}}$, one can also view $L_p(1-m,\chi)$ in a finite extension of $\mathbb{Q}_p$ to obtain a p-adic L-function, cf. [29, Ch. 5]. Thereby one has to choose a fixed embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$, the completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Here we keep the focus on functions on $\mathbb{Z}_p$.

Proposition 7.6. *Let $\chi = 1$ or χ be a primitive quadratic character* $(\mathrm{mod}\ \mathfrak{f}_\chi)$*. Assume that $p > 3$ in case $\chi = 1$, otherwise $p^e \neq \mathfrak{f}_\chi$, $e \geq 1$. Let $l \in 2\mathbb{N}_0$, where $0 \leq l < \varphi(q)$ and $l \neq \delta_\chi$. Then $L_{p,l}(\cdot,\chi)$ can be uniquely extended to $\mathbb{Z}_p$ such that $L_{p,l}(\cdot,\chi) \in \mathcal{K}_{p,2}$.*

Proof. The conditions above satisfy Theorem 7.5, which we use in a weaker form to obtain

$$\nabla^n L_{p,l}(0,\chi) = \nabla_{\varphi(q)}^n L_p(1-s,\chi)\,\Big|_{s=\delta_\chi+l} \equiv 0 \pmod{p^n}$$

for all $n \geq 0$. Thus $L_{p,l}(\cdot,\chi)$ is a Kummer function and Proposition 7.2 gives the result. □

It remains the case $l = \delta_\chi = 0$ and $\chi \neq 1$. Here we have the situation that $L_{p,l}(0,\chi)$ is not defined. In spite of that $L_{p,l}(s,\chi)$ can be uniquely extended to $\mathbb{Z}_p$ by removing the discontinuity at $s = 0$.

Proposition 7.7. *Let χ be an even primitive quadratic character* $(\mathrm{mod}\ \mathfrak{f}_\chi)$*. Assume that $p^e \neq \mathfrak{f}_\chi$, $e \geq 1$. Then $L_{p,0}(\cdot,\chi)$ can be uniquely extended to $\mathbb{Z}_p$ such that $L_{p,0}(s,\chi) \in \mathcal{K}_{p,2}$ with a removable discontinuity at $s = 0$.*

Proof. By Theorem 7.5 we have

$$\nabla^n L_{p,0}(1,\chi) = \nabla_{\varphi(q)}^n L_p(1-s,\chi)\,\Big|_{s=\varphi(q)} \equiv 0 \pmod{p^n}$$

for all $n \geq 0$. Define $f(s) = L_{p,l}(s+1,\chi)$ for $s \in \mathbb{N}_0$. Then f is a Kummer function and Proposition 7.2 shows that $f \in \mathcal{K}_{p,2}$. Now define $L_{p,l}(s,\chi) = f(s-1)$ for $s \in \mathbb{Z}_p$. Consequently we get an extended function $L_{p,l}(s,\chi) \in \mathcal{K}_{p,2}$, which is defined at $s = 0$. □

Henceforth we regard the functions $L_{p,l}(\cdot,\chi)$ as p-adic functions lying in $\mathcal{K}_{p,2}$ as a result of Propositions 7.6 and 7.7.

The definition of irregular primes and irregular pairs, which is usually introduced in the context of Bernoulli numbers and the class number of cyclotomic fields, was generalized to generalized Bernoulli numbers by several authors, cf. Ernvall [7], Hao and Parry [14], and Holden [15], who explicitly studied irregular primes over real quadratic fields.

Our definition of χ-irregular primes differs somewhat, such that we associate a χ-irregular pair (p, l) with a p-adic L-function $L_{p,l}(\cdot,\chi)$, where l is always even. Moreover, we exclude primes that divide the conductor of χ, which are considered separately.

Definition 7.8. *Let $\mathfrak{X}_2$ be the set of primitive quadratic characters* $(\mathrm{mod}\ \mathfrak{f}_\chi)$ *including $\chi = 1$. We define for $\chi \in \mathfrak{X}_2$ the set of χ-irregular primes by*

$$\Psi_\chi^{\mathrm{irr}} = \{(p,l) : L_{p,l}(0,\chi) \in p\mathbb{Z}_p, p > 3, p \nmid \mathfrak{f}_\chi, 2 \le l \le p-3, 2 \mid l\}.$$

The index of χ-irregularity of p is defined to be

$$i_\chi(p) = \#\{(p,l) \in \Psi_\chi^{\mathrm{irr}} : 2 \le l \le p-3, 2 \mid l\}.$$

The aim of this new definition is that we have a correspondence between

$$(p,l) \in \Psi_\chi^{\mathrm{irr}} \qquad \longleftrightarrow \qquad L_{p,l}(\cdot,\chi) \in \mathcal{K}_{p,2}^0,$$

which enables us to study the behavior of the χ-irregular prime p and its powers as divisors of $L_{p,l}(\cdot,\chi)$. Collecting information about all $(p,l) \in \Psi_\chi^{\mathrm{irr}}$, we then achieve a description of the structure of the underlying L-function at negative integer arguments.

Theorem 7.9. *Let $\chi \in \mathfrak{X}_2$ and $n \in 2\mathbb{N}$. Set $s_{p,l} = s_{p,l}(n)$. Then*

$$\left|L(1-(\delta_\chi+n),\chi)\right|_\infty = \mathfrak{I}(n,\chi)\,\mathfrak{S}(n,\chi)\,\mathfrak{D}(n,\chi),$$

where

$$\mathfrak{I}(n,\chi) = \prod_{\substack{(p,l)\in\Psi_\chi^{\mathrm{irr}} \\ l\equiv n \pmod{p-1}}} \left|L_{p,l}(s_{p,l},\chi)\right|_p^{-1},$$

$$\mathfrak{S}(n,\chi) = \prod_{\substack{p\mid\mathfrak{f}_\chi \\ l\equiv n \pmod{p-1}}} \left|L_{p,l}(s_{p,l},\chi)\right|_p^{-1},$$

$$\mathfrak{D}(n,\chi) = \prod_{\substack{p\nmid\mathfrak{f}_\chi \\ p-1\mid n}} \left|L_{p,0}(s_{p,0},\chi)\right|_p^{-1}.$$

Moreover, if χ is odd, then

$$\mathfrak{D}(n,\chi) = \prod_{\substack{\chi(p)=1 \\ p-1\mid n}} \left|L_{p,0}(s_{p,0},\chi)\right|_p^{-1} \prod_{\substack{\chi(p)=-1 \\ p\mid 2B_{1,\chi} \\ p-1\mid n}} \left|L_{p,0}(s_{p,0},\chi)\right|_p^{-1}.$$

Proof. Note that $L(1-(\delta_\chi+n),\chi) \in \mathbb{Q}^*$. The product formula states that

$$\prod_{p\in\mathbb{P}\cup\{\infty\}} \left|L(1-(\delta_\chi+n),\chi)\right|_p = 1.$$

Since $\delta_\chi + n \ge 2$, we have for all primes p that

$$\left|L(1-(\delta_\chi+n),\chi)\right|_p = \left|L_p(1-(\delta_\chi+n),\chi)\right|_p = \left|L_{p,l}(s_{p,l},\chi)\right|_p,$$

where $l \equiv n \pmod{p-1}$. Further we divide $\mathbb{P}$ into the disjoint sets $I_1 = \{p : p \mid \mathfrak{f}_\chi\}$, $I_2 = \{p : p \nmid \mathfrak{f}_\chi, p-1 \mid n\}$, and $I_3 = \{p : p \nmid \mathfrak{f}_\chi, p-1 \nmid n\}$. Thus

$$|L(1-(\delta_\chi+n),\chi)|_\infty = \prod_{\substack{p \in I_1 \cup I_2 \cup I_3 \\ l \equiv n \pmod{p-1}}} |L_{p,l}(s_{p,l},\chi)|_p^{-1}.$$

We can split the above product into three products over I_1, I_2, and I_3. The product over I_1, resp., I_2 equals $\mathfrak{S}(n,\chi)$, resp., $\mathfrak{D}(n,\chi)$. It remains the product over I_3. We have to show that

$$\prod_{\substack{p \nmid \mathfrak{f}_\chi,\, l \neq 0 \\ l \equiv n \pmod{p-1}}} |L_{p,l}(s_{p,l},\chi)|_p^{-1} = \prod_{\substack{(p,l) \in \Psi_\chi^{\mathrm{irr}} \\ l \equiv n \pmod{p-1}}} |L_{p,l}(s_{p,l},\chi)|_p^{-1}.$$

By Proposition 7.6 the left product above consists only of functions $L_{p,l}(\cdot,\chi) \in \mathcal{K}_{p,2}$. From Definition 7.8 we deduce for these functions that $(p,l) \notin \Psi_\chi^{\mathrm{irr}}$ implies $L_{p,l}(\cdot,\chi) \in \mathcal{K}_{p,2}^*$.

Now, assume that χ is odd. Regarding the product of $\mathfrak{D}(n,\chi)$ we can also write $I_2 = I_2^+ \cup I_2^-$, where $I_2^\pm = \{p : \chi(p) = \pm 1, p-1 \mid n\}$. If $p \in I_2^-$, then we have

$$L_{p,0}(0,\chi) = -(1-\chi(p))B_{1,\chi} = -2B_{1,\chi} \neq 0.$$

Here we use the non-trivial fact that $B_{1,\chi} \neq 0$ for odd χ, cf. [29, Thm. 4.9, p. 38]. Since $L_{p,0}(\cdot,\chi) \in \mathcal{K}_{p,2}$, it follows that $L_{p,0}(\cdot,\chi) \in \mathcal{K}_{p,2}^*$ when $p \nmid 2B_{1,\chi}$. □

Our main interest is focused on the product of $\mathfrak{I}(\cdot,\chi)$, since the χ-irregular primes and their powers are the fundamental elements, building mainly the values of $L(\cdot,\chi)$ at negative integer arguments. Theorem 7.9 shows that

$$\mathfrak{I}(n,\chi) = \prod_{\substack{(p,l) \in \Psi_\chi^{\mathrm{irr}} \\ l \equiv n \pmod{p-1}}} |L_{p,l}(s_{p,l},\chi)|_p^{-1}, \quad n \in 2\mathbb{N},$$

where the functions, lying in $\mathcal{K}_{p,2}$, can have a different behavior, such that

$$|L_{p,l}(s_{p,l},\chi)|_p^{-1} \text{ is } \begin{cases} \text{constant}, \\ \text{bounded}, \\ \text{unbounded}, \end{cases}$$

as a result of the last section. As supported by computations, mainly functions of $\widehat{\mathcal{K}}_{p,2}^0$ have been found. Thus, powers of χ-irregular primes, unbounded as $n \to \infty$ for $n \in 2\mathbb{N}$, seem to give contribution to the values of $L(1-(\delta_\chi+n),\chi)$.

Remark 7.10. Buhler et al [1] calculated irregular pairs and cyclotomic invariants for all primes below 12 million. Due to their results, we deduce for these pairs that $\zeta_{p,l} \in \widehat{\mathcal{K}}_{p,2}^0$, $p^2 \nmid \zeta_{p,l}(0)$, and the zero $\xi \in \mathbb{Z}_p^*$ by Proposition 6.21. Holden [15] showed that there are examples of χ-irregular pairs, χ a primitive quadratic character, such that $p^2 \mid L_{p,l}(0,\chi)$. However, we have recalculated these examples to demonstrate that the functions in question have always $\lambda_f = 1$ and lie in $\widehat{\mathcal{K}}_{p,2}^0$; consequently the zero $\xi \in p\mathbb{Z}_p$ by Proposition 6.22. These and further computational results are given in [18]; see also Example 7.17.

These aspects lead to the following conjecture about the *irregular part* of the values of $L(\cdot,\chi)$ at negative integer arguments.

Conjecture 7.11. *Assume the conditions of Theorem 7.9. We postulate the following conjecture, which may hold either in weak or strong form, for a given L-function.*

(1) *Weak form:*

$$\mathfrak{I}(n,\chi) = \prod_{\substack{(p,l)\in\Psi_\chi^{\mathrm{irr}} \\ l\equiv n \pmod{p-1}}} p^{\lambda_f} \prod_{\nu=1}^{\lambda_f} \left| s_{p,l} - \xi_{p,l}^{(\nu)} \right|_p^{-1}, \quad n \in 2\mathbb{N},$$

where $\xi_{p,l}^{(\nu)} \in \mathbb{Z}_p$ are the roots of the corresponding function $f = L_{p,l}(\cdot,\chi) \in \mathcal{K}_{p,2}^{\mathrm{s}}$ with $\lambda_f < p$.

(2) *Strong form:*

$$\mathfrak{I}(n,\chi) = \prod_{\substack{(p,l)\in\Psi_\chi^{\mathrm{irr}} \\ l\equiv n \pmod{p-1}}} p \left| s_{p,l} - \xi_{p,l} \right|_p^{-1}, \quad n \in 2\mathbb{N},$$

where $\xi_{p,l} \in \mathbb{Z}_p$ is the root of the corresponding function $L_{p,l}(\cdot,\chi) \in \widehat{\mathcal{K}}_{p,2}^{0}$.

Clearly, products of L-functions provide examples, trivially take $L(\cdot,\chi)^2$, where we then get a conjectural product representation as above in *weak form*. Therefore we define a product of L-functions, which only makes sense when the characters have the same parity. Otherwise we would get a zero function at negative integer arguments.

Definition 7.12. *Let $\chi_1, \chi_2 \in \mathfrak{X}_2$. Assume that χ_1 and χ_2 have the same parity. Define the product $L(\cdot, \chi_1 \otimes \chi_2) = L(\cdot,\chi_1) L(\cdot,\chi_2)$, which transfers to all other definitions. Further define $\Psi_{\chi_1,\chi_2}^{\mathrm{irr}} = \Psi_{\chi_1}^{\mathrm{irr}} \cap \Psi_{\chi_2}^{\mathrm{irr}}$.*

Proposition 7.13. *Let $\chi_1, \chi_2 \in \mathfrak{X}_2$ having the same parity. If $\Psi_{\chi_1,\chi_2}^{\mathrm{irr}} \neq \emptyset$, then there exist functions $f = L_{p,l}(\cdot, \chi_1 \otimes \chi_2) \in \mathcal{K}_{p,2}$ with $(p,l) \in \Psi_{\chi_1,\chi_2}^{\mathrm{irr}}$ and $\lambda_f \geq 2$. Assuming Conjecture 7.11, these functions lie in $\mathcal{K}_{p,2}^{\mathrm{s}}$ and $\mathfrak{I}(n, \chi_1 \otimes \chi_2)$ has a weak form.*

Proof. We use the notation of Theorem 7.9. Let $n \in 2\mathbb{N}$. We then obtain

$$\mathfrak{I}(n, \chi_1 \otimes \chi_2) = \mathfrak{I}(n,\chi_1)\, \mathfrak{I}(n,\chi_2) = \mathfrak{I}_1\, \mathfrak{I}_2\, \mathfrak{I}_3,$$

where

$$\mathfrak{I}_\nu = \prod_{\substack{(p,l)\in\Psi_{\chi_\nu}^{\mathrm{irr}} \setminus \Psi_{\chi_1,\chi_2}^{\mathrm{irr}} \\ l\equiv n \pmod{p-1}}} \left| L_{p,l}(s_{p,l}, \chi_\nu) \right|_p^{-1}, \quad \nu = 1, 2,$$

$$\mathfrak{I}_3 = \prod_{\substack{(p,l)\in\Psi_{\chi_1,\chi_2}^{\mathrm{irr}} \\ l\equiv n \pmod{p-1}}} \left| L_{p,l}(s_{p,l}, \chi_1) L_{p,l}(s_{p,l}, \chi_2) \right|_p^{-1}.$$

If $\Psi_{\chi_1,\chi_2}^{\mathrm{irr}} \neq \emptyset$, then the product of $\mathfrak{I}_3$ cannot be trivial for all n. For $(p,l) \in \Psi_{\chi_1,\chi_2}^{\mathrm{irr}}$ we get

$$L_{p,l}(\cdot,\chi_1) L_{p,l}(\cdot,\chi_2) = L_{p,l}(\cdot, \chi_1 \otimes \chi_2) \in \mathcal{K}_{p,2}.$$

Since both functions $L_{p,l}(\cdot,\chi_\nu) \in \mathcal{K}^0_{p,2}$, the product $f = L_{p,l}(\cdot,\chi_1 \otimes \chi_2)$ has $\lambda_f \geq 2$ by Proposition 6.10. Assuming Conjecture 7.11, all these functions lie in $\mathcal{K}^{\mathrm{s}}_{p,2}$ and so do their products. Since we have $\lambda_f \geq 2$ for some functions, $\mathfrak{I}(n,\chi_1 \otimes \chi_2)$ has a weak form. $\square$

Next, we consider the connection with the Dedekind zeta function.

Definition 7.14. *Let K be an algebraic number field. The Dedekind zeta function is defined by*

$$\zeta_K(z) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-z}, \quad z \in \mathbb{C},\ \mathrm{Re}\, z > 1,$$

where the sum runs over all nonzero integral ideals of K and $N(\mathfrak{a})$ denotes the norm of the ideal $\mathfrak{a}$.

We need the well known decomposition for quadratic fields, cf. [29, Thm. 4.3, p. 34].

Theorem 7.15. *Let D be the fundamental discriminant of the quadratic field $K = \mathbb{Q}(\sqrt{D})$. Then*

$$\zeta_K(z) = \zeta(z) L(z,\chi_D), \quad z \in \mathbb{C},$$

where $\chi_D(\cdot) = \left(\frac{D}{\cdot}\right)$ is the Kronecker symbol.

Corollary 7.16. *Let $D > 0$ be the fundamental discriminant of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$. Then*

$$\zeta_K(1-n) = L(1-n, 1 \otimes \chi_D), \quad n \geq 2.$$

The irregular part of $\zeta_K(1-n)$ is described by $\mathfrak{I}(n, 1 \otimes \chi_D)$ for $n \in 2\mathbb{N}$.

Proof. This follows by Proposition 7.13, since $\chi = 1$ and χ_D have the same parity. $\square$

Holden [15] studied ζ_K in case of real quadratic fields in context of χ_D-irregular primes and their distribution. We use here his computational results, which we have recalculated and extended for our purpose. He found an example for $D = 77$ such that $(37,32)$ is both an irregular pair and a χ_D-irregular pair; this implies $\Psi^{\mathrm{irr}}_{1,\chi_D} \neq \emptyset$. As mentioned earlier in Remark 7.10, there are examples where $p^2 \mid L_{p,l}(0,\chi_D)$, e.g., for $D = 5$ and $(p,l) = (443,216)$. Holden mentions that there are other examples, particularly for $D = 5$ and $p < 50$, but without publishing these data. Therefore we have used the tables of χ_D-irregular primes for $D = 5$ of Hao and Parry [14], to find the example for $D = 5$ and $(p,l) = (19,8)$, which is more suitable for our calculations below. The computations are reported in [18].

Example 7.17. Computed zeros ξ and fixed points τ $\pmod{p^{10}}$.

(1) Case $D = 5$, $(p,l) = (19,8)$, $f = L_{p,l}(\cdot,\chi_D) \in \widehat{\mathcal{K}}^0_{p,2}$:

f	values $(s = 0,1)$ / p-adic digits $s_0,\ldots,s_9$
Δ_f, λ_f	16, 1
$\mathrm{ord}_p\, f(s)$	2, 1
ξ	0, 7, 18, 11, 12, 10, 10, 8, 14, 0
τ	0, 0, 2, 13, 11, 4, 15, 6, 12, 16

(2) Case $D = 77$, $(p, l) = (37, 32)$, $f = L_{p,l}(\cdot, 1 \otimes \chi_D) \in \mathcal{K}^{\mathrm{s}}_{p,2}$:

f	values $(s = 0, 1)$ / p-adic digits $s_0, \ldots, s_9$
Δ_f, λ_f	0, 2
$\operatorname{ord}_p f(s)$	2, 2
ξ_1	7, 28, 21, 30, 4, 17, 26, 13, 32, 35
ξ_{χ_D}	9, 36, 26, 31, 25, 30, 21, 36, 30, 33
τ	0, 0, 14, 35, 13, 27, 30, 3, 22, 29

Remark. Surely, there are several authors, who already computed zeros of p-adic L-functions, mostly in context of Iwasawa theory. These calculations were performed by searching a start solution $\pmod{p^r}$ for some $r \geq 1$ and further using Newton's method. In contrast, we give here a necessary and sufficient condition ($f(0) \in p\mathbb{Z}_p$ and $\Delta_f \neq 0$, or $\lambda_f = 1$ for both), so that $f \in \widehat{\mathcal{K}}^0_{p,2}$ which shows the existence of a zero and $|f(s)|_p$ reduces to a linear term (for $p = 2$ we also need the condition $2 \mid \Delta_f(2)$).

At the end, we consider the *non-irregular part* $\mathfrak{D}(\cdot, \chi)$ in case χ is odd. Here we have the interesting situation, that the functions $L_{p,0}(s, \chi)$ have a zero at $s = 0$ when $\chi(p) = 1$.

Definition 7.18. *Let $\chi \in \mathfrak{X}_2$ where χ is odd. We define*

$$\Psi^{\mathrm{exc}}_\chi = \{(p, 0) : L_{p,0}(1, \chi) \in p^2\mathbb{Z}_p, p > 3, \chi(p) = 1\}$$

as the set of χ-exceptional pairs. We further define for $(p, 0) \in \Psi^{\mathrm{exc}}_\chi$ the functions

$$\tilde{L}_{p,0}(s, \chi) = \begin{cases} L_{p,0}(s, \chi)/ps, & s \neq 0, \\ L'_{p,0}(s, \chi)/p, & s = 0, \end{cases} \qquad s \in \mathbb{Z}_p.$$

Remark. As a result of Proposition 5.12 and Lemma 4.11, we have for $p > 3$ that

$$\tilde{L}_{p,0}(\cdot, \chi) \in \mathcal{K}^{\mathrm{d}}_{p,1} \quad \text{and} \quad \tilde{L}_{p,0}(0, \chi) = \int_{\mathbb{Z}_p} \eth L_{p,0}(s, \chi)\, ds.$$

The value of $\tilde{L}_{p,0}(0, \chi)$ is easily computable by (4.6) of Lemma 4.11.

The next theorem shows that Definition 7.18 is well defined. The set Ψ^{exc}_χ can be seen as an analogue to Ψ^{irr}_χ, where all functions $f = L_{p,0}(\cdot, \chi)$ with $(p, 0) \in \Psi^{\mathrm{exc}}_\chi$ have the property that $\lambda_f > 1$.

Theorem 7.19. *Let $\chi \in \mathfrak{X}_2$ where χ is odd. Let $p > 3$ where $\chi(p) = 1$. For $f = L_{p,0}(\cdot, \chi)$ we have the following statements:*

1. $\lambda_f = 1 \iff \operatorname{ord}_p L_{p,0}(1, \chi) = 1 \iff (p, 0) \notin \Psi^{\mathrm{exc}}_\chi$.
2. $\lambda_f > 1 \iff \operatorname{ord}_p L_{p,0}(1, \chi) \geq 2 \iff (p, 0) \in \Psi^{\mathrm{exc}}_\chi$.
3. *If $\lambda_f = 1$, then $|L_{p,0}(s, \chi)|_p = |ps|_p$ for $s \in \mathbb{Z}_p$.*
4. *If $\lambda_f = 2$, then $|L_{p,0}(s, \chi)|_p = |p^2 s(s - \xi_{p,0})|_p$ for $s \in \mathbb{Z}_p$, where $\xi_{p,0}$ is the unique simple zero of $\tilde{L}_{p,0}(\cdot, \chi) \in \mathcal{K}^{\mathrm{d}}_{p,1}$.*

Proof. (1)-(3): Since χ is odd, we have

$$L_{p,0}(0,\chi) = -(1-\chi(p))B_{1,\chi} = 0.$$

Therefore $L_{p,0}(\cdot,\chi)$ has a zero at $s=0$. By Lemma 4.11 and Definition 7.18 we obtain

$$L_{p,0}(s,\chi) = ps\,\tilde{L}_{p,0}(s,\chi), \quad s \in \mathbb{Z}_p.$$

Using (4.7) of Lemma 4.11, we have

$$\tilde{L}_{p,0}(s,\chi) \equiv \Delta_f \pmod{p\mathbb{Z}_p}.$$

If $\lambda_f = 1$, then $|\tilde{L}_{p,0}(s,\chi)|_p = 1$ and $|L_{p,0}(s,\chi)|_p = |ps|_p$ for $s \in \mathbb{Z}_p$, thus $\operatorname{ord}_p L_{p,0}(1,\chi) = 1$. Conversely, $\lambda_f > 1$ implies that $|\tilde{L}_{p,0}(s,\chi)|_p < 1$ for $s \in \mathbb{Z}_p$ and $p^2 \mid L_{p,0}(1,\chi)$.

(4): Since $L_{p,0}(s,\chi)$ has a zero at $s=0$ and $\lambda_f = 2$, this function satisfies the conditions of Theorem 5.15. It follows that $|L_{p,0}(s,\chi)|_p = |p^2 s(s-\xi_{p,0})|_p$ for $s \in \mathbb{Z}_p$ and $\xi_{p,0}$ is the unique simple zero of $\tilde{L}_{p,0}(\cdot,\chi)$. Proposition 5.12 shows that $\tilde{L}_{p,0}(\cdot,\chi) \in \mathcal{K}^{\mathrm{d}}_{p,1}$. □

We achieve a more detailed decomposition of $\mathfrak{D}(n,\chi)$ as follows.

Theorem 7.20. *Assume the conditions of Theorem 7.9 where χ is odd. We have*

$$\mathfrak{D}(n,\chi) = \mathfrak{D}_{2,3}(n,\chi)\,\mathfrak{D}_+(n,\chi)\,\mathfrak{D}_-(n,\chi)\,\mathfrak{D}_0(n,\chi),$$

where

$$\mathfrak{D}_{2,3}(n,\chi) = \prod_{p \in I_\chi^{2,3}} |L_{p,0}(s_{p,0},\chi)|_p^{-1}, \quad I_\chi^{2,3} = \{p \le 3 : \chi(p) \ne 0, p \mid (1-\chi(p))B_{1,\chi}\},$$

$$\mathfrak{D}_+(n,\chi) = \prod_{\substack{p \in I_\chi^+ \\ p-1 \mid n}} |pn|_p^{-1}, \quad I_\chi^+ = \{p > 3 : \chi(p) = 1\},$$

$$\mathfrak{D}_-(n,\chi) = \prod_{\substack{p \in I_\chi^- \\ p-1 \mid n}} |L_{p,0}(s_{p,0},\chi)|_p^{-1}, \quad I_\chi^- = \{p > 3 : \chi(p) = -1, p \mid B_{1,\chi}\},$$

$$\mathfrak{D}_0(n,\chi) = \prod_{\substack{(p,0) \in \Psi_\chi^{\mathrm{exc}} \\ p-1 \mid n}} |\tilde{L}_{p,0}(s_{p,0},\chi)|_p^{-1}.$$

Proof. By Theorem 7.9 we have

$$\mathfrak{D}(n,\chi) = \prod_{\substack{\chi(p)=1 \\ p-1 \mid n}} |L_{p,0}(s_{p,0},\chi)|_p^{-1} \prod_{\substack{\chi(p)=-1 \\ p \mid 2B_{1,\chi} \\ p-1 \mid n}} |L_{p,0}(s_{p,0},\chi)|_p^{-1},$$

where all functions lie in $\mathcal{K}_{p,2}$. First we separate the factors for $p=2$ and $p=3$ from the products above. This defines $\mathfrak{D}_{2,3}(n,\chi)$ and $I_\chi^{2,3}$, where we use the original condition $p \mid L_{p,0}(0,\chi) = -(1-\chi(p))B_{1,\chi}$, such that $L_{p,0}(\cdot,\chi) \notin \mathcal{K}^*_{p,2}$. The second product above for $p > 3$ defines $\mathfrak{D}_-(n,\chi)$, which covers the case $\chi(p) = -1$ and the modified condition

$p \mid B_{1,\chi}$. Now, we consider the remaining case $\chi(p) = 1$ for $p > 3$. We use Theorem 7.19 to obtain

$$\prod_{\substack{p>3 \\ \chi(p)=1 \\ p-1 \mid n}} |L_{p,0}(s_{p,0}, \chi)|_p^{-1} = \mathfrak{D}_+(n, \chi)\, \mathfrak{D}_0(n, \chi),$$

where we have used that

$$|L_{p,0}(s_{p,0}, \chi)|_p^{-1} = |p s_{p,0}|_p^{-1} = |pn/(p-1)|_p^{-1} = |pn|_p^{-1}, \quad (p, 0) \notin \Psi_\chi^{\mathrm{exc}},$$

and

$$|L_{p,0}(s_{p,0}, \chi)|_p^{-1} = |pn|_p^{-1}\, |\tilde{L}_{p,0}(s_{p,0}, \chi)|_p^{-1}, \quad (p, 0) \in \Psi_\chi^{\mathrm{exc}}. \qquad \square$$

We examine now examples of χ-exceptional primes. Let χ_{-3} be the non-principal character (mod 3) associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$. Ernvall [7] studied the χ-irregular pairs of the so-called D-numbers, which are given by

$$D_n = 3L(-n, \chi_{-3}), \quad n \geq 1.$$

He states that $p^2 \mid D_{p-1}$ occurs only for $p = 13$, 181, and 2521 below 10^4, where he also remarks that the primes $p = 13$ and $p = 181$ were already found by Ferrero [10]. We have found exactly two further primes below 10^6: $p = 76543$ and $p = 489061$. The condition $p^2 \mid D_{p-1}$ is equivalent to $L_{p,0}(1, \chi_{-3}) \in p^2\mathbb{Z}_p$ and therefore $(p, 0) \in \Psi_{\chi_{-3}}^{\mathrm{exc}}$ for these five primes. Certainly, these primes p satisfy $\chi_{-3}(p) = 1$ or equivalently $p \equiv 1 \pmod 3$.

Regarding the primes mentioned above, the following table shows that these functions $f = L_{p,0}(\cdot, \chi_{-3})$ have always $\lambda_f = 2$; the corresponding functions $\tilde{L}_{p,0}(\cdot, \chi_{-3})$ have each time a unique simple zero by Theorem 7.19. More results are given in [18].

Table 7.21. Computed parameters of functions $f = L_{p,0}(\cdot, \chi_{-3}) \in \mathcal{K}_{p,2}^2$:

p	λ_f	$\mathrm{ord}_p\, f(1)$	$\mathrm{ord}_p\, \Delta_f(2)$
13	2	2	0
181	2	2	0
2521	2	2	0
76543	2	≥ 2	0
489061	2	≥ 2	0

Using Algorithm 5.7, we have computed the zero of the δ-degenerate function $\tilde{L}_{p,0}(\cdot, \chi_{-3})$ for $p = 13$.

Example 7.22. Computed zero $\xi \pmod{p^{10}}$ of $f = \tilde{L}_{p,0}(\cdot, \chi_{-3}) \in \mathcal{K}_{p,1}^{\mathrm{d}}$ for the case $p = 13$:

f	values / p-adic digits $s_0, \ldots, s_9$
Δ_f, λ_f	3, 1
ξ	3, 8, 2, 11, 1, 1, 10, 12, 7, 1

The functions $\tilde{L}_{p,0}(\cdot, \chi)$ seem to behave like the functions of the irregular part. In contrast, these functions lie in $\mathcal{K}_{p,1}^{\mathrm{d}}$ having a defect in their Mahler expansion. Therefore

we raise the following conjecture about the functions $\tilde{L}_{p,0}(\cdot,\chi)$ in the case of χ-exceptional pairs.

Conjecture 7.23. *Assume the conditions of Theorem 7.20. Then*

$$\mathfrak{D}_0(n,\chi) = \prod_{\substack{(p,0)\in\Psi_\chi^{\mathrm{exc}} \\ p-1\mid n}} \left| p\left(s_{p,0}-\xi_{p,0}\right)\right|_p^{-1}, \quad n \in 2\mathbb{N},$$

where $\xi_{p,0}$ is the zero of $\tilde{L}_{p,0}(\cdot,\chi) \in \mathcal{K}_{p,1}^{\mathrm{d}}$.

8. Bernoulli and Euler numbers

The Bernoulli and Euler numbers are defined by

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi,$$

$$\frac{2}{e^z+e^{-z}} = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!}, \quad |z| < 2\pi.$$

The numbers B_n are rational, whereas the numbers E_n are integers. It easily follows that

$$\begin{aligned} E_n &= -2\frac{B_{n+1,\chi_{-4}}}{n+1} = 2L(-n,\chi_{-4}), \quad n \geq 0, \\ B_n &= B_{n,1} \qquad\qquad = -n\zeta(1-n), \quad n \geq 2. \end{aligned}$$

We also have the connection with the Dedekind zeta function of $\mathbb{Q}(i)$ that

$$\zeta_{\mathbb{Q}(i)}(z) = \zeta(z)L(z,\chi_{-4}), \quad z \in \mathbb{C}.$$

In 1850 Kummer introduced congruences about Bernoulli and Euler numbers in the following form, which have been greatly generalized after that and now are called Kummer congruences. For the sake of completeness we cite Kummer's theorem.

Theorem 8.1 (Kummer [21])**.** *Let n, r be positive integers, where n is even.*

(1) *If $p-1 \nmid n$ and $n > r$, then*

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^{r-\nu} \frac{B_{n+\nu(p-1)}}{n+\nu(p-1)} \equiv 0 \pmod{p^r}.$$

(2) *If $n > r$, then*

$$\sum_{\nu=0}^{r} \binom{r}{\nu} (-1)^{r-\nu} E_{n+\nu(p-1)} \equiv 0 \pmod{p^r}.$$

Theorem 8.1, formulated with Euler factors to remove the restriction $n > r$, would already be sufficient to define Kummer functions, which, extended to $\mathbb{Z}_p$, lie in $\mathcal{K}_{p,2}$. Now, one can apply all results about $\mathcal{K}_{p,2}$ to B_n/n and E_n, always modified by an Euler factor, since the last section has shown that $\zeta_{p,l}$ and $L_{p,l}(\cdot,\chi_{-4})$ are functions of $\mathcal{K}_{p,2}$. Proposition 3.10 enables us to compute B_n/n and $E_n \pmod{p^r}$ for arbitrary even integers n. Algorithm 4.6, resp., Algorithm 4.8 shows how to compute a zero, resp., a fixed point $\pmod{p^r}$ of $\zeta_{p,l}$ and $L_{p,l}(\cdot,\chi_{-4})$. As an application, we can sharpen the usual Kummer congruences for the Bernoulli numbers for those cases where the converse also holds.

Proposition 8.2 (Strong Kummer congruences)**.** *Let $p > 3$ and $l \in 2\mathbb{N}$ where $0 < l < p-1$. Set $\varepsilon_l = 1 - p$ in case $l = 2$, otherwise $\varepsilon_l = 1$. Then*

$$\frac{B_{l+p-1}}{l+p-1} \not\equiv \varepsilon_l \frac{B_l}{l} \pmod{p^2}$$

if and only if

$$n \equiv m \pmod{\varphi(p^r)} \quad \Longleftrightarrow \quad (1-p^{n-1})\frac{B_n}{n} \equiv (1-p^{m-1})\frac{B_m}{m} \pmod{p^r}$$

for $n, m \in 2\mathbb{N}$ such that $n \equiv m \equiv l \pmod{p-1}$ and $1 \le r \le 1 + \operatorname{ord}_p(n-m)$.

Proof. We observe that $\nabla\zeta_{p,l}(0) \equiv \varepsilon_l B_l/l - B_{l+p-1}/(l+p-1) \pmod{p^2\mathbb{Z}_p}$, where the Euler factors vanish except for $l = 2$ and $\zeta_{p,l}(0)$. Therefore the condition above is equivalent to $\Delta_{\zeta_{p,l}} \neq 0$ to ensure that $\zeta_{p,l} \in \widehat{\mathcal{K}}_{p,2}$. By Corollary 3.8 we then have

$$|p\,(s-t)|_p = |\zeta_{p,l}(s) - \zeta_{p,l}(t)|_p\,, \quad s,t \in \mathbb{Z}_p.$$

Conversely, $\Delta_{\zeta_{p,l}} = 0$ implies that

$$|p\,(s-t)|_p > |\zeta_{p,l}(s) - \zeta_{p,l}(t)|_p\,, \quad s,t \in \mathbb{Z}_p,$$

as a result of Proposition 3.7. Transferring this back to the Bernoulli numbers with $n = s(p-1)+l, m = t(p-1)+l \in \mathbb{N}$ gives the result. □

Remark. One cannot omit the Euler factor for $l = 2$ in the condition of the proposition above. For example, if $p = 13$, then $B_{14}/14 = B_2/2$, but $B_{14}/14 - (1-p)B_2/2 = 13/12 \not\equiv 0 \pmod{p^2}$. Without the condition, we only have the implication "$\Rightarrow$", which equals the usual Kummer congruences.

Example 8.3. Computed zeros ξ and fixed points $\tau \pmod{p^{10}}$ of functions of $\widehat{\mathcal{K}}^0_{p,2}$.

(1) Case $(p,l) = (37,32)$ and $f = \zeta_{p,l} \in \widehat{\mathcal{K}}^0_{p,2}$:

f	values / p-adic digits $s_0,\dots,s_9$
Δ_f, λ_f	16, 1
ξ	7, 28, 21, 30, 4, 17, 26, 13, 32, 35
τ	0, 36, 28, 6, 26, 35, 27, 23, 10, 11

It follows that the smallest indices for $\operatorname{ord}_p(B_{n_\nu}/n_\nu) = \nu$ are, e.g., $n_1 = 32$, $n_2 = 284$, $n_3 = 37\,580$, $n_4 = 1\,072\,544$, and $n_5 = 55\,777\,784$.

(2) Case $(p,l) = (19,10)$ and $f = L_{p,l}(\cdot, \chi_{-4}) \in \widehat{\mathcal{K}}^0_{p,2}$:

f	values / p-adic digits $s_0, \ldots, s_9$
Δ_f, λ_f	5, 1
ξ	17, 6, 13, 18, 17, 10, 6, 18, 12, 14
τ	0, 10, 8, 17, 15, 1, 4, 9, 14, 18

It follows that the smallest indices for $\operatorname{ord}_p E_{n_\nu} = \nu$ are, e.g., $n_1 = 10$, $n_2 = 316$, $n_3 = 2\,368$, $n_4 = 86\,842$, and $n_5 = 2\,309\,158$.

Definition 7.8 of χ-irregular primes for $\chi = 1$, resp., $\chi = \chi_{-4}$ agrees with the usual definition of irregular primes regarding B_n, resp., E_n. The latter are often called E-irregular primes, cf. Carlitz [3], Ernvall and Metsänkylä [8]. As a result of Carlitz [3], there are infinitely many irregular primes regarding B_n and E_n. Equivalently, Jensen [16] showed a more special result for the Bernoulli numbers before, that there are infinitely many irregular primes $p \equiv 3 \pmod 4$. Ernvall [7] later showed that there are infinitely many E-irregular primes $p \not\equiv \pm 1 \pmod 8$. Therefore $\#\Psi_1^{\mathrm{irr}} = \infty$ and $\#\Psi_{\chi_{-4}}^{\mathrm{irr}} = \infty$.

We will derive a conjectural formula for the structure of the Bernoulli and Euler numbers. Recall the notations of Theorem 7.9, which we use in the following. Considering Remark 7.10 and Conjecture 7.11, we may assume that the corresponding products $\mathfrak{I}(\cdot, 1)$ and $\mathfrak{I}(\cdot, \chi_{-4})$ fulfil the strong form. First, we consider the Bernoulli numbers, where we need the famous fact about their denominator.

Theorem 8.4 (von Staudt [27], Clausen [5])**.** *Let $n \in 2\mathbb{N}$. Then*

$$B_n + \sum_{p-1 \mid n} \frac{1}{p} \in \mathbb{Z},$$

which implies that the denominator of B_n equals $\prod_{p-1 \mid n} p$.

Proposition 8.5. *We have*

$$\mathfrak{S}(n,1) = 1 \quad \text{and} \quad \mathfrak{D}(n,1) = \prod_{p-1 \mid n} |pn|_p \,, \qquad n \in 2\mathbb{N}.$$

Proof. We make use of Theorem 7.9. Since the conductor $\mathfrak{f}_1 = 1$, the product of $\mathfrak{S}(n,1)$ is always trivial. By Definition 7.4 and Theorem 8.4, we obtain

$$\mathfrak{D}(n,1) = \prod_{p-1 \mid n} |\zeta_{p,0}(s_{p,0})|_p^{-1} = \prod_{p-1 \mid n} \left| \frac{B_n}{n} \right|_p^{-1} = \prod_{p-1 \mid n} |pn|_p \,. \qquad \square$$

Combining Theorem 7.9, Proposition 8.5, and Conjecture 7.11, we deduce the following.

Conjecture 8.6. *The structure of the Bernoulli numbers is given by*

$$\left| \frac{B_n}{n} \right|_\infty = \prod_{p-1 \mid n} |pn|_p \prod_{\substack{(p,l) \in \Psi_1^{\mathrm{irr}} \\ l \equiv n \pmod{p-1}}} |p\,(s_{p,l} - \xi_{p,l})|_p^{-1} \,, \qquad n \in 2\mathbb{N},$$

where $\xi_{p,l}$ is the zero of $\zeta_{p,l}$.

Remark. This conjecture about the Bernoulli numbers was already given by the author, see [17, Rem. 4.17, p. 421], where the numerator of B_n/n, assuming the conjecture, can be described by zeros and the denominator of B_n/n can be described, without any assumption, by poles of p-adic zeta functions. Since the Bernoulli numbers and the Riemann zeta function can be viewed as the prototype of the generalized Bernoulli numbers and L-functions, one may speculate, whether this *simple* formula above holds generally.

Secondly, we consider the Euler numbers. Here we have the more complicated behavior of the non-irregular part, since we do not have a denominator as in the case of the Bernoulli numbers.

Theorem 8.7 (Frobenius [12][1], Carlitz [2][2], [4][3])**.** *Let $n \in 2\mathbb{N}$. Then*

$$E_n \equiv \begin{cases} 0, & (p \equiv 1 \pmod 4), \\ 2, & (p \equiv 3 \pmod 4), \end{cases} \pmod{p^r} \quad {}^{1,2} \tag{8.1}$$

when $\varphi(p^r) \mid n$. Moreover,

$$\frac{E_n}{2} = -\frac{B_{n+1,\chi_{-4}}}{n+1} \equiv \frac{1}{2} \pmod 1. \quad {}^{3} \tag{8.2}$$

We can sharpen the result of the theorem above by determining the exact p-power, that divides E_n in (8.1), for most cases.

Proposition 8.8. *Let $p \equiv 1 \pmod 4$ and $n \in 2\mathbb{N}$ where $p-1 \mid n$. We have the following statements:*

(1) *If $\operatorname{ord}_p E_{p-1} = 1$, then*

$$|E_n|_p = |pn|_p\,, \quad \textit{resp.,} \quad \operatorname{ord}_p E_n = 1 + \operatorname{ord}_p n.$$

Otherwise, we have

$$|E_n|_p < |pn|_p\,, \quad \textit{resp.,} \quad \operatorname{ord}_p E_n \geq 2 + \operatorname{ord}_p n.$$

(2) *If $\operatorname{ord}_p E_{p-1} \geq 2$ and $\operatorname{ord}_p(E_{2(p-1)} - 2E_{p-1}) = 2$, then*

$$|E_n|_p = \left|p^2 s(s-\xi)\right|_p,$$

where $s = n/(p-1)$ and $\xi \in \mathbb{Z}_p$ is the unique simple zero of $\tilde{L}_{p,0}(\cdot, \chi_{-4})$.

Proof. Note that $p > 3$. We apply Theorem 7.19 to $f = L_{p,0}(\cdot, \chi_{-4})$, where we have the connection with the Euler numbers by

$$|L_{p,0}(s, \chi_{-4})|_p = \left|\tfrac{1}{2} E_{s(p-1)}\right|_p = \left|E_{s(p-1)}\right|_p, \quad s \in \mathbb{N},$$

and especially $|L_{p,0}(1, \chi_{-4})|_p = |E_{p-1}|_p$. Moreover, we have

$$L_{p,0}(s, \chi_{-4}) = ps\, \tilde{L}_{p,0}(s, \chi_{-4}), \quad s \in \mathbb{Z}_p.$$

(1): Since

$$\operatorname{ord}_p E_{p-1} = 1 \iff \operatorname{ord}_p L_{p,0}(1, \chi_{-4}) = 1 \iff \lambda_f = 1,$$

we obtain $|L_{p,0}(s,\chi_{-4})|_p = |ps|_p$ for $s \in \mathbb{Z}_p$. It follows for $n = s(p-1) \in \mathbb{N}$ that

$$|E_n|_p = |L_{p,0}(s,\chi_{-4})|_p = |ps|_p = |pn/(p-1)|_p = |pn|_p \,.$$

Otherwise, we have the case that $\lambda_f > 1$, $\operatorname{ord}_p E_{p-1} \geq 2$, and $|\tilde{L}_{p,0}(s,\chi_{-4})|_p < 1$ for $s \in \mathbb{Z}_p$. This implies the inequalities given above.

(2): We show that the conditions $\operatorname{ord}_p E_{p-1} \geq 2$ and $\operatorname{ord}_p(E_{2(p-1)} - 2E_{p-1}) = 2$ give the necessary and sufficient conditions for $\lambda_f = 2$. As seen above, $\operatorname{ord}_p E_{p-1} \geq 2$ implies that $\lambda_f > 1$. Thus it remains the condition $\Delta_f(2) = \nabla^2 L_{p,0}(0,\chi_{-4})/p^2 \in \mathbb{Z}_p^*$ to ensure that $\lambda_f = 2$. Since $L_{p,0}(0,\chi_{-4}) = 0$, we obtain the condition $\operatorname{ord}_p(L_{p,0}(2,\chi_{-4}) - 2L_{p,0}(1,\chi_{-4})) = 2$, which is equivalent to $\operatorname{ord}_p(E_{2(p-1)} - 2E_{p-1}) = 2$, since the Euler factors have no effect. Now, we have established that $\lambda_f = 2$. Hence

$$|E_n|_p = |L_{p,0}(s,\chi_{-4})|_p = \left|p^2 s(s-\xi)\right|_p ,$$

where $n = s(p-1) \in \mathbb{N}$ and $\xi \in \mathbb{Z}_p$ is the unique simple zero of $\tilde{L}_{p,0}(\cdot,\chi_{-4})$. $\square$

Remark. Ernvall [7] notices that the exception such that $p^2 \mid E_{p-1}$ occurs first for $p = 29\,789$, which is the only known example for the Euler numbers yet. Therefore $(29\,789, 0) \in \Psi^{\mathrm{exc}}_{\chi_{-4}}$. The table below represents our computations for this prime showing that $\tilde{L}_{p,0}(\cdot,\chi_{-4})$ has a unique simple zero. Moreover, using a congruence of [8, p. 628], which also follows as a special case of Theorem 9.6 for the functions $T_{p,l}(\cdot,\chi)$ for $\chi = \chi_{-4}$, we have not found any further exceptional prime $p \equiv 1 \pmod 4$ below 10^6. These exceptional primes seem to be very rarely in the case of the Euler numbers. The result of [8] is the computation of cyclotomic invariants and E-irregular pairs below 10^4. We have extended the computations of E-irregular pairs up to $p < 10^5$ to show that $L_{p,l}(\cdot,\chi_{-4}) \in \widehat{\mathcal{K}}^0_{p,2}$ for $(p,l) \in \Psi^{\mathrm{irr}}_{\chi_{-4}}$ in this range. We give all details of the computations and used methods in [18].

Table 8.9. Computed parameters of function $f = L_{p,0}(\cdot,\chi_{-4}) \in \mathcal{K}_{p,2}$ for $p = 29\,789$:

p	λ_f	$\operatorname{ord}_p f(1)$	$\operatorname{ord}_p \Delta_f(2)$
29 789	2	2	0

verified by

	$\pmod{p^3}$	$\pmod{p^3}/p^2$	ord_p
E_{p-1}	22651377283046	25526	2
$E_{2(p-1)}$	20830464245954	23474	2
$E_{2(p-1)} - 2E_{p-1}$	1962007175931	2211	2

Proposition 8.10. *We have*

$$\mathfrak{S}(n,\chi_{-4}) = \frac{1}{2} \quad \textit{and} \quad \mathfrak{D}(n,\chi_{-4}) = \mathfrak{D}_0(n,\chi_{-4}) \prod_{\substack{p-1 \mid n \\ p \equiv 1 \pmod 4}} |pn|_p^{-1} , \qquad n \in 2\mathbb{N}.$$

Proof. We apply Theorems 7.9, 7.20, and 8.7. Since $\mathfrak{f}_{\chi_{-4}} = 4$, we obtain by (8.2) that

$$\mathfrak{S}(n,\chi_{-4}) = |L_{2,0}(s_{2,0},\chi_{-4})|_2^{-1} = \left|\frac{E_n}{2}\right|_2^{-1} = \frac{1}{2}.$$

Observing that χ_{-4} is odd and $2B_{1,\chi_{-4}} = -E_0 = -1$, the product of $\mathfrak{D}(n,\chi_{-4})$ reduces to

$$\mathfrak{D}_0(n,\chi_{-4})\mathfrak{D}_+(n,\chi_{-4}) = \mathfrak{D}_0(n,\chi_{-4}) \prod_{\substack{p-1 \mid n \\ p \equiv 1 \pmod 4}} |pn|_p^{-1},$$

where we have used that $\chi_{-4}(p) = 1 \iff p \equiv 1 \pmod 4$. □

Note that the factor $\mathfrak{S}(n,\chi_{-4}) = \frac{1}{2}$ is cancelled regarding $E_n = 2L(-n,\chi_{-4})$ for even n. Combining Theorems 7.9 and 7.20, Proposition 8.10, and Conjectures 7.11 and 7.23, we deduce the following.

Conjecture 8.11. *The structure of the Euler numbers is given by*

$$|E_n|_\infty = \prod_{\substack{p-1 \mid n \\ p \equiv 1 \pmod 4}} |pn|_p^{-1} \prod_{\substack{(p,l) \in \Psi^{\mathrm{irr}}_{\chi_{-4}} \cup \Psi^{\mathrm{exc}}_{\chi_{-4}} \\ l \equiv n \pmod{p-1}}} |p\,(s_{p,l} - \xi_{p,l})|_p^{-1}, \quad n \in 2\mathbb{N},$$

where $\xi_{p,l}$ is the zero of $L_{p,l}(\cdot,\chi_{-4})$ when $l \neq 0$ and $\xi_{p,0}$ is the zero of $\tilde{L}_{p,0}(\cdot,\chi_{-4})$.

Remark. It is quite remarkable that the Bernoulli and Euler numbers, defined by simple generating functions and lying in $\mathbb{Q}$, resp., $\mathbb{Z}$, can be completely described only with the aid of the theory of p-adic functions. In other words, the Riemann zeta function and L-functions at negative integer arguments encode p-adic information about zeros of p-adic functions.

9. Fermat quotients

In this section, let p be an odd prime. Let $\mathcal{U}_p = 1 + p\mathbb{Z}_p$. As usual, we have the decomposition

$$a = \omega(a)\,\langle a \rangle\,, \quad a \in \mathbb{Z}_p^*,$$

where $\langle a \rangle \in \mathcal{U}_p$ and $\omega : \mathbb{Z}_p^* \to \mathbb{Z}_p^*$ is the Teichmüller character, that gives the $(p-1)$-th roots of unity in $\mathbb{Q}_p$ and satisfies $\omega(a) \equiv a \pmod{p\mathbb{Z}_p}$. Define the Fermat quotient by

$$q(a) = \frac{a^{p-1}-1}{p}, \quad a \in \mathbb{Z}_p^*.$$

Further define

$$u(a) : \mathbb{Z}_p^* \to \mathcal{U}_p, \quad a \mapsto a^{p-1}.$$

Alternatively, we also have $u(a) = \langle a \rangle^{p-1} = 1 + p\,q(a)$. Basic properties of the Fermat quotient were given by Lerch [23]; one of these is the similar behavior like the log function:

$$q(ab) \equiv q(a) + q(b) \pmod{p\mathbb{Z}_p}, \quad a,b \in \mathbb{Z}_p^*.$$

Now, we introduce the functions $T_{p,l}$ as follows.

Definition 9.1. *Let l, r be integers where $r \geq 1$. We define*

$$T_{p,l}^r(s) = \sum_{a=1}^{p-1} a^l q(a)^r u(a)^s, \quad s \in \mathbb{Z}_p.$$

We also write $T_{p,l}(s)$ for $T_{p,l}^1(s)$.

Proposition 9.2. *Let ν, l, r be integers where $\nu \geq 0$ and $r \geq 1$. We have the following statements:*

(1) $T_{p,l}^r \in \mathcal{K}_{p,2}$.
(2) $\eth^\nu \, T_{p,l}^r = T_{p,l}^{r+\nu}$.
(3) *The Mahler expansion is given by*

$$T_{p,l}^r(s) = \sum_{\nu \geq 0} \Delta_{T_{p,l}^r}(\nu)\, p^\nu \binom{s}{\nu}, \quad s \in \mathbb{Z}_p,$$

where $\Delta_{T_{p,l}^r}(\nu) = T_{p,l}^{r+\nu}(0)$.

(4) *We have*

$$T_{p,l}^r(s) \equiv T_{p,l+p-1}^r(s) \equiv T_{p,l}^{r+p-1}(s) \pmod{p\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.$$

(5) *We have a certain recurrent behavior of the coefficients such that*

$$\Delta_{T_{p,l}^r}(\nu) \equiv \Delta_{T_{p,l}^r}(\nu + p - 1) \equiv \Delta_{T_{p,l+p-1}^r}(\nu) \equiv \Delta_{T_{p,l}^{r+p-1}}(\nu) \pmod{p\mathbb{Z}_p}.$$

(6) *Define $S_n(m) = 1^n + \cdots + (m-1)^n$ for $m \geq 1$, $n \geq 0$. If $l \geq 0$, then*

$$T_{p,l}^r(s) = p^{-r}\, \nabla_{p-1}^r S_t(p) \,\Big|_{t=l+s(p-1)}, \quad s \in \mathbb{N}_0.$$

Proof. Let $s \in \mathbb{Z}_p$. (1): From Proposition 6.2 we deduce that the functions

$$a^l q(a)^r u(a)^s \in \mathcal{K}_{p,2}$$

for all $a \in \{1, \ldots, p-1\}$. Thus the sum $T_{p,l}^r(s) \in \mathcal{K}_{p,2}$. (2): A simple calculation shows that

$$\eth\, u(a)^s = \nabla u(a)^s / p = q(a) u(a)^s.$$

So we get $\eth\, T_{p,l}^r(s) = T_{p,l}^{r+1}(s)$ and iteratively $\eth^\nu \, T_{p,l}^r(s) = T_{p,l}^{r+\nu}(s)$. (3): This follows by (2) and Lemma 3.4. (4),(5): This is a consequence of Fermat's little theorem. The case (5) follows by $\Delta_{T_{p,l}^r}(\nu) = T_{p,l}^{r+\nu}(0)$. (6): Using the binomial expansion

$$a^l q(a)^r u(a)^s = p^{-r} a^{l+s(p-1)} (a^{p-1} - 1)^r = p^{-r}\, \nabla_{p-1}^r a^t \,\Big|_{t=l+s(p-1)}$$

and summing over a yield the result. $\square$

The next theorem shows a link between the behavior of the functions $T_{p,l}$ and $\zeta_{p,l}$.

Theorem 9.3. *Let $p > 3$ and $l \in 2\mathbb{N}$ where $0 < l < p-1$. We have the relations*

$$\begin{aligned} T_{p,l}(s) &\equiv \zeta_{p,l}(s) \pmod{p\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p, \\ \Delta_{T_{p,l}} &\equiv 2\Delta_{\zeta_{p,l}} \pmod{p\mathbb{Z}_p}, \quad l \neq 2. \end{aligned}$$

The functions $T_{p,l}$ and $\zeta_{p,l}$ have the same classification in $\mathcal{K}_{p,2}$ such that

$$\begin{aligned} T_{p,l} \in \mathcal{K}^*_{p,2} &\iff \zeta_{p,l} \in \mathcal{K}^*_{p,2}, \\ T_{p,l} \in \widehat{\mathcal{K}}^0_{p,2} &\iff \zeta_{p,l} \in \widehat{\mathcal{K}}^0_{p,2}. \end{aligned}$$

Proof. We use the well known congruences, giving a relation between Fermat quotients and Bernoulli numbers, which can be deduced by Proposition 9.2 (6), cf. [9, Prop. 1,2, p. 855]:

$$T^1_{p,l}(0) = \sum_{a=1}^{p-1} a^l q(a) \equiv -\frac{B_l}{l} \pmod{p\mathbb{Z}_p}, \tag{9.1}$$

$$T^2_{p,l}(0) = \sum_{a=1}^{p-1} a^l q(a)^2 \equiv -\frac{2}{p}\left(\frac{B_{l+p-1}}{l+p-1} - \frac{B_l}{l}\right) \pmod{p\mathbb{Z}_p}, \quad l \neq 2. \tag{9.2}$$

From (9.1) it follows that

$$T_{p,l}(0) = T^1_{p,l}(0) \equiv \zeta_{p,l}(0) \pmod{p\mathbb{Z}_p},$$

since the Euler factor of $\zeta_{p,l}$ vanishes for $l \geq 2$. This also shows that $T_{p,l}(s) \equiv \zeta_{p,l}(s) \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$ and consequently that $T_{p,l} \in \mathcal{K}^*_{p,2} \iff \zeta_{p,l} \in \mathcal{K}^*_{p,2}$. Because $\frac{B_2}{2} = \frac{1}{12} \not\equiv 0 \pmod{p\mathbb{Z}_p}$, we always have $T_{p,2}, \zeta_{p,2} \in \mathcal{K}^*_{p,2}$. Similarly we deduce by (9.2) and Proposition 9.2 (2), (3) that

$$\Delta_{T_{p,l}} \equiv \eth\, T_{p,l}(0) = T^2_{p,l}(0) \equiv 2\eth\, \zeta_{p,l}(0) \equiv 2\Delta_{\zeta_{p,l}} \pmod{p\mathbb{Z}_p},$$

observing that the Euler factors of $\eth\,\zeta_{p,l}$ vanish for $l > 2$. Now, we have $T_{p,l} \notin \mathcal{K}^*_{p,2} \iff \zeta_{p,l} \notin \mathcal{K}^*_{p,2}$, which implies $l > 2$. In these cases we also have $\Delta_{T_{p,l}} \equiv 2\Delta_{\zeta_{p,l}} \pmod{p\mathbb{Z}_p}$ and consequently $T_{p,l} \in \widehat{\mathcal{K}}^0_{p,2} \iff \zeta_{p,l} \in \widehat{\mathcal{K}}^0_{p,2}$. □

We can generalize the results in the following way.

Definition 9.4. *Let $\chi \in \mathfrak{X}_2$, $\chi \neq 1$, and $p \nmid \mathfrak{f}_\chi$. Let l, r be integers where $r \geq 1$. We define*

$$T^r_{p,l}(s,\chi) = \frac{1}{\mathfrak{f}_\chi} \sum_{\substack{a=1 \\ (a,p)=1}}^{p\,\mathfrak{f}_\chi} \chi(a) a^{l+\delta_\chi} q(a)^r u(a)^s, \quad s \in \mathbb{Z}_p.$$

We write $T_{p,l}(s,\chi)$ for $T^1_{p,l}(s,\chi)$. Further define

$$S_{n,\chi}(m) = \sum_{a=1}^{m} \chi(a) a^n, \quad S^*_{n,\chi}(m) = \sum_{\substack{a=1 \\ (a,p)=1}}^{m} \chi(a) a^n, \quad m \geq 1, n \geq 0.$$

The generalized Bernoulli polynomials are given by

$$B_{n,\chi}(x) = \sum_{\nu=0}^{n} \binom{n}{\nu} B_{\nu,\chi}\, x^{n-\nu}, \quad n \geq 1, x \in \mathbb{R}.$$

Proposition 9.5. *Let $\chi \in \mathfrak{X}_2$, $\chi \neq 1$, and $p \nmid \mathfrak{f}_\chi$. Let ν, l, r be integers where $\nu \geq 0$ and $r \geq 1$. We have the following statements:*

(1) $T_{p,l}^{r}(\cdot, \chi) \in \mathcal{K}_{p,2}$.
(2) $\eth^{\nu}\, T_{p,l}^{r}(\cdot, \chi) = T_{p,l}^{r+\nu}(\cdot, \chi)$.
(3) *The Mahler expansion is given by*

$$T_{p,l}^{r}(s, \chi) = \sum_{\nu \geq 0} \Delta_{T_{p,l}^{r}(\cdot,\chi)}(\nu)\, p^{\nu} \binom{s}{\nu}, \quad s \in \mathbb{Z}_p,$$

where $\Delta_{T_{p,l}^{r}(\cdot,\chi)}(\nu) = T_{p,l}^{r+\nu}(0, \chi)$.
(4) *We have*

$$T_{p,l}^{r}(s, \chi) \equiv T_{p,l+p-1}^{r}(s, \chi) \equiv T_{p,l}^{r+p-1}(s, \chi) \pmod{p\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.$$

(5) *We have a certain recurrent behavior of the coefficients such that*

$$\begin{aligned} \Delta_{T_{p,l}^{r}(\cdot,\chi)}(\nu) &\equiv \Delta_{T_{p,l}^{r}(\cdot,\chi)}(\nu + p - 1) \\ &\equiv \Delta_{T_{p,l+p-1}^{r}(\cdot,\chi)}(\nu) \equiv \Delta_{T_{p,l}^{r+p-1}(\cdot,\chi)}(\nu) \pmod{p\mathbb{Z}_p}. \end{aligned}$$

(6) *If $l \geq 0$, then*

$$T_{p,l}^{r}(s, \chi) = \mathfrak{f}_\chi^{-1} p^{-r}\, \nabla_{p-1}^{r} S_{t,\chi}^{*}(p\, \mathfrak{f}_\chi)\,\Big|_{\,t = l + \delta_\chi + s(p-1)}, \quad s \in \mathbb{N}_0.$$

Proof. The proof is exactly derived as the proof of Proposition 9.2 by considering the additional factor $\chi(a)$ in the sum of $T_{p,l}^{r}$ and excluding those a where $p \mid a$. $\square$

Theorem 9.6. *Let $\chi \in \mathfrak{X}_2$, $\chi \neq 1$, $p > 3$, and $p \nmid \mathfrak{f}_\chi$. Let $l \in 2\mathbb{N}_0$ where $0 \leq l < p - 1$. We have the relations*

$$\begin{aligned} T_{p,l}(s, \chi) &\equiv L_{p,l}(s, \chi) \pmod{p\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p, \\ \Delta_{T_{p,l}(\cdot,\chi)} &\equiv 2\Delta_{L_{p,l}(\cdot,\chi)} \pmod{p\mathbb{Z}_p}. \end{aligned}$$

The functions $T_{p,l}(\cdot, \chi)$ and $L_{p,l}(\cdot, \chi)$ have the same classification in $\mathcal{K}_{p,2}$ such that

$$\begin{aligned} T_{p,l}(\cdot, \chi) \in \mathcal{K}_{p,2}^{*} \quad &\iff \quad L_{p,l}(\cdot, \chi) \in \mathcal{K}_{p,2}^{*}, \\ T_{p,l}(\cdot, \chi) \in \widehat{\mathcal{K}}_{p,2}^{0} \quad &\iff \quad L_{p,l}(\cdot, \chi) \in \widehat{\mathcal{K}}_{p,2}^{0}. \end{aligned}$$

The case $\chi = 1$ is compatible with the former results of Proposition 9.2 and Theorem 9.3 since $T_{p,l}^{r}(s, 1) = T_{p,l}^{r}(s)$. The difference between the cases $\chi = 1$ and $\chi \neq 1$ is only caused by the von Staudt-Clausen Theorem 8.4, while the latter case implies that we already have p-integrality of the numbers $B_{n,\chi}$ in question. We need some preparations to prove Theorem 9.6.

Proposition 9.7 ([25, p. 463][1], [4][2])**.** *Let $\chi \neq 1$ be a primitive non-principal character where $p \nmid \mathfrak{f}_\chi$ and $m, n \geq 1$. We have the following statements:*

(1)
$$S_{n,\chi}(m) = \frac{1}{n+1}\left(B_{n+1,\chi}(m) - B_{n+1,\chi}\right), \quad \mathfrak{f}_\chi \mid m. \quad ^1$$

(2)
$$B_{0,\chi} = 0, \quad B_{n,\chi}/n \text{ is } p\text{-integral.} \quad ^2$$

(3)
$$S^*_{n,\chi}(m) = S_{n,\chi}(m) - \chi(p)p^n\, S_{n,\chi}(m/p), \quad p \mid m.$$

(4) *If $p\,\mathfrak{f}_\chi \mid m$ and $n \equiv \delta_\chi \pmod 2$, then*
$$\begin{aligned} S_{n,\chi}(m)/m &\equiv B_{n,\chi} && \pmod{p^2}, \quad n \geq 1, \\ S_{n,\chi}(m)/m &\equiv B_{n,\chi} + \binom{n}{3}\frac{B_{n-2,\chi}}{n-2}\, m^2 && \pmod{p^4}, \quad n \geq 3. \end{aligned}$$

Proof. (3): This follows by
$$S_{n,\chi}(m) = \sum_{\substack{a=1 \\ (a,p)=1}}^{m} \chi(a)a^n + \sum_{a=1}^{m/p} \chi(pa)(pa)^n.$$

(4): By Definition 9.4 and using (1) and (2), we obtain
$$S_{n,\chi}(m)/m = \frac{1}{n+1}\sum_{\nu=0}^{n}\binom{n+1}{\nu} B_{\nu,\chi}\, m^{n-\nu} = \sum_{\nu=1}^{n}\binom{n}{\nu-1}\frac{B_{\nu,\chi}}{\nu}\, m^{n-\nu},$$
where the numbers $B_{\nu,\chi}/\nu$ are p-integral. Since $n \equiv \delta_\chi \pmod 2$, we have $B_{n,\chi} \neq 0$ and $B_{n-1,\chi} = 0$; also $B_{n-2,\chi} \neq 0$ and $B_{n-3,\chi} = 0$ if $n \geq 3$. By assumption $p \mid m$ and this implies the congruences in $\mathbb{Q}(\chi)$. □

Lemma 9.8. *Let $\chi \neq 1$ be a primitive non-principal character where $p > 3$ and $p \nmid \mathfrak{f}_\chi$. Let n, r be integers with $n, r \geq 1$ and $n \equiv \delta_\chi \pmod 2$. We have*
$$\nabla^r_{p-1}\,(1-\chi(p)p^{t-1})B_{t,\chi}\,\Big|_{t=n} \equiv -r\,\nabla^{r-1}_{p-1}\,(1-\chi(p)p^{t-1})\frac{B_{t,\chi}}{t}\,\Big|_{t=n} \pmod{p^r}.$$
Moreover the congruence above vanishes $\pmod{p^{r-1}}$.

Proof. Note that the congruences are valid in $\mathbb{Q}(\chi)$. For brevity we write $\widetilde{B}_{t,\chi} = (1 - \chi(p)p^{t-1})B_{t,\chi}/t$. As a consequence of the Kummer congruences, cf. Theorem 7.5, we have
$$\nabla^r_{p-1}\,\widetilde{B}_{t,\chi}\,\Big|_{t=n} \equiv 0 \pmod{p^r}. \tag{9.3}$$

Thus we can write

$$\begin{aligned}
&\nabla^r_{p-1}\,(1-\chi(p)p^{t-1})B_{t,\chi}\,\big|_{\,t=n} \\
&\quad\equiv \nabla^r_{p-1}\left(t\widetilde{B}_{t,\chi} - (n+r(p-1))\widetilde{B}_{t,\chi}\right)\,\big|_{\,t=n} \\
&\quad\equiv \sum_{\nu=0}^{r}\binom{r}{\nu}(-1)^{r-\nu}(-(r-\nu)(p-1))\widetilde{B}_{n+\nu(p-1),\chi} \\
&\quad\equiv r(p-1)\sum_{\nu=0}^{r-1}\binom{r-1}{\nu}(-1)^{r-1-\nu}\widetilde{B}_{n+\nu(p-1),\chi} \\
&\quad\equiv -r\,\nabla^{r-1}_{p-1}\,(1-\chi(p)p^{t-1})\frac{B_{t,\chi}}{t}\,\big|_{\,t=n} \pmod{p^r},
\end{aligned}$$

where we have used that $\binom{r}{\nu} = \frac{r}{r-\nu}\binom{r-1}{\nu}$ and the last part of the congruences is divisible by p^{r-1} in view of (9.3). $\square$

Proof of Theorem 9.6. Set $l' = l + \delta_\chi$ and $m = p\,\mathfrak{f}_\chi$. Note that $\mathfrak{f}_\chi \in \mathbb{Z}_p^*$, $l' \geq 0$, and $p \geq 5$. Let $r \geq 1$. We write $\varepsilon_t = (1-\chi(p)p^{t-1})$ for the Euler factors. By Propositions 9.5 (6), 9.7 (3) we obtain

$$\begin{aligned}
p^{r-1}\,T^r_{p,l}(s,\chi) &= \nabla^r_{p-1}S^*_{t,\chi}(m)/m\,\big|_{\,t=l'+s(p-1)} \\
&= \nabla^r_{p-1}S_{t,\chi}(m)/m\,\big|_{\,t=l'+s(p-1)} \\
&\quad - \chi(p)\,\nabla^r_{p-1}p^{t-1}S_{t,\chi}(\mathfrak{f}_\chi)/\mathfrak{f}_\chi\,\big|_{\,t=l'+s(p-1)}, \quad s \in \mathbb{N}_0.
\end{aligned} \tag{9.4}$$

To avoid complemental Euler factors in the cases where $s = 0$ and l' is small, caused by the second summand above, we shall shift the index l' to $l'_1 = l' + p - 1$. This simplifies the congruences and we can add Euler factors when needed. Recall Corollary 5.11 which shows that the coefficients $\Delta_f(\nu)$ of functions $f \in \mathcal{K}_{p,2}$ are invariant (mod $p\mathbb{Z}_p$) under translation.

Case $r = 1$: Using (9.4), Propositions 9.5, 9.7 (4), and Lemma 9.8, we deduce that

$$\begin{aligned}
T_{p,l}(0,\chi) \equiv T_{p,l}(1,\chi) &\equiv \nabla_{p-1}S_{t,\chi}(m)/m\,\big|_{\,t=l'_1} \\
&\equiv \nabla_{p-1}B_{t,\chi}\,\big|_{\,t=l'_1} \equiv \nabla_{p-1}\,\varepsilon_t B_{t,\chi}\,\big|_{\,t=l'_1} \\
&\equiv -\varepsilon_{l'_1}\frac{B_{l'_1,\chi}}{l'_1} \equiv L_{p,l}(1,\chi) \equiv L_{p,l}(0,\chi) \pmod{p\mathbb{Z}_p}.
\end{aligned}$$

Since $T_{p,l}(\cdot,\chi), L_{p,l}(\cdot,\chi) \in \mathcal{K}_{p,2}$, we also have $T_{p,l}(s,\chi) \equiv L_{p,l}(s,\chi) \pmod{p\mathbb{Z}_p}$ for $s \in \mathbb{Z}_p$. This shows that $T_{p,l}(\cdot,\chi) \in \mathcal{K}^*_{p,2} \iff L_{p,l}(\cdot,\chi) \in \mathcal{K}^*_{p,2}$.

Case $r = 2$: First, we have by Proposition 9.5 that

$$\Delta_{T_{p,l}(\cdot,\chi)} \equiv \eth\,T_{p,l}(0,\chi) \equiv T^2_{p,l}(0,\chi) \equiv T^2_{p,l}(1,\chi) \pmod{p\mathbb{Z}_p}.$$

Again, using (9.4), Proposition 9.7 (4), and Lemma 9.8, we derive that

$$\begin{aligned} p\,T_{p,l}^2(1,\chi) &\equiv \nabla_{p-1}^2 S_{t,\chi}(m)/m\,\Big|_{t=l_1'} \\ &\equiv \nabla_{p-1}^2 B_{t,\chi}\,\Big|_{t=l_1'} \equiv \nabla_{p-1}^2\, \varepsilon_t B_{t,\chi}\,\Big|_{t=l_1'} \\ &\equiv -2\nabla_{p-1}\,\varepsilon_t \frac{B_{t,\chi}}{t}\,\Big|_{t=l_1'} \pmod{p^2\mathbb{Z}_p} \end{aligned}$$

and further that

$$\begin{aligned} T_{p,l}^2(1,\chi) &\equiv -\frac{2}{p}\nabla_{p-1}\,\varepsilon_t \frac{B_{t,\chi}}{t}\,\Big|_{t=l_1'} \\ &\equiv 2\eth L_{p,l}(1,\chi) \equiv 2\eth L_{p,l}(0,\chi) \equiv 2\Delta_{L_{p,l}(\cdot,\chi)} \pmod{p\mathbb{Z}_p}. \end{aligned}$$

Thus $\Delta_{T_{p,l}(\cdot,\chi)} \equiv 2\Delta_{L_{p,l}(\cdot,\chi)} \pmod{p\mathbb{Z}_p}$. Considering case $r=1$ we deduce that $T_{p,l}(\cdot,\chi) \in \widehat{\mathcal{K}}_{p,2}^0 \iff L_{p,l}(\cdot,\chi) \in \widehat{\mathcal{K}}_{p,2}^0$. $\square$

Remark. Let $\chi \in \mathfrak{X}_2$, $p \nmid \mathfrak{f}_\chi$, $l \in 2\mathbb{N}$, and $0 < l < p-1$. The function $T_{p,l}(\cdot,\chi)$ has a unique simple zero if and only if $L_{p,l}(\cdot,\chi)$ has a unique simple zero. This can only happen, when $(p,l) \in \Psi_\chi^{\text{irr}}$ and $\Delta_{T_{p,l}(\cdot,\chi)} \equiv 2\Delta_{L_{p,l}(\cdot,\chi)} \not\equiv 0 \pmod{p\mathbb{Z}_p}$. Example 9.10 shows the analogy to Example 8.3. Moreover, we have a kind of reciprocity relation as follows.

Corollary 9.9. *Let $\chi \in \mathfrak{X}_2$, $p \nmid \mathfrak{f}_\chi$, $l \in 2\mathbb{N}$, $0 < l < p-1$, and $(p,l) \in \Psi_\chi^{\text{irr}}$. If $T_{p,l}(\cdot,\chi) \in \widehat{\mathcal{K}}_{p,2}^0$ or $L_{p,l}(\cdot,\chi) \in \widehat{\mathcal{K}}_{p,2}^0$, then*

$$\frac{\tau_T}{\tau_L} \cdot \frac{\xi_L}{\xi_T} \equiv 2 \pmod{p\mathbb{Z}_p},$$

where τ_T, $\tau_L \in p\mathbb{Z}_p$ is the fixed point and ξ_T, $\xi_L \in \mathbb{Z}_p$ is the zero of $T_{p,l}(\cdot,\chi)$, $L_{p,l}(\cdot,\chi)$, respectively.

Proof. This follows by Lemma 4.9, Theorem 9.3 for $\chi = 1$, and Theorem 9.6 for $\chi \neq 1$. $\square$

Example 9.10. Computed zeros ξ and fixed points $\tau \pmod{p^{10}}$ of functions of $\widehat{\mathcal{K}}_{p,2}^0$.

(1) Case $(p,l) = (37,32)$ and $f = T_{p,l} \in \widehat{\mathcal{K}}_{p,2}^0$:

f	values / p-adic digits $s_0, \ldots, s_9$
Δ_f, λ_f	32, 1
ξ	19, 1, 24, 12, 16, 24, 22, 26, 12, 33
τ	0, 21, 31, 31, 14, 25, 15, 2, 10, 27

(2) Case $(p,l) = (19,10)$ and $f = T_{p,l}(\cdot,\chi_{-4}) \in \widehat{\mathcal{K}}_{p,2}^0$:

f	values / p-adic digits $s_0, \ldots, s_9$
Δ_f, λ_f	10, 1
ξ	4, 8, 6, 1, 18, 14, 8, 3, 3, 3
τ	0, 17, 8, 16, 0, 2, 7, 2, 10, 14

Theorem 9.11. *Let $\chi \in \mathfrak{X}_2$, $\chi \neq 1$, $p > 3$, and $p \nmid \mathfrak{f}_\chi$. Let $l \in 2\mathbb{N}_0$ where $0 \leq l < p-1$. We have the relations*

$$\begin{aligned} 3\Delta_{L_{p,l}(\cdot,\chi)}(2) &\equiv \Delta_{T_{p,l}(\cdot,\chi)}(2) - \mathfrak{f}_\chi^2\, \Delta_{T_{p,l-2}(\cdot,\chi)}(0) \\ &\equiv T_{p,l}^3(0,\chi) - \mathfrak{f}_\chi^2\, T_{p,l-2}(0,\chi) \qquad \pmod{p\mathbb{Z}_p}, \\ 4\Delta_{L_{p,l}(\cdot,\chi)}(3) &\equiv \Delta_{T_{p,l}(\cdot,\chi)}(3) - 2\mathfrak{f}_\chi^2\, \Delta_{T_{p,l-2}(\cdot,\chi)} \\ &\equiv T_{p,l}^4(0,\chi) - 2\mathfrak{f}_\chi^2\, T_{p,l-2}^2(0,\chi) \qquad \pmod{p\mathbb{Z}_p}. \end{aligned}$$

Proof. Set $l' = l + \delta_\chi$ and $m = p\,\mathfrak{f}_\chi$. Define $\ell_2 = p-3$ for $l = 0$ otherwise $\ell_2 = l-2$. Note that $\mathfrak{f}_\chi \in \mathbb{Z}_p^*$, $l' \geq 0$, and $p \geq 5$. We write $\varepsilon_t = (1-\chi(p)p^{t-1})$. We use the same arguments given in the proof of Theorem 9.6. Thus we shift the index l' to $l_2' = l' + 2(p-1)$, which is sufficient for the following congruences since $p \geq 5$. We now consider the cases $r = 3$ and $r = 4$ simultaneously. By Proposition 9.5 we have

$$\Delta_{T_{p,l}(\cdot,\chi)}(r-1) \equiv \eth^{r-1}\, T_{p,l}(0,\chi) \equiv T_{p,l}^r(0,\chi) \equiv T_{p,l}^r(2,\chi) \pmod{p\mathbb{Z}_p}.$$

From (9.4) and Proposition 9.7 (4) we deduce that

$$\begin{aligned} p^{r-1}\, T_{p,l}^r(2,\chi) &\equiv \nabla_{p-1}^r S_{t,\chi}(m)/m \,\Big|_{t=l_2'} \\ &\equiv \nabla_{p-1}^r\, \varepsilon_t B_{t,\chi} \,\Big|_{t=l_2'} \\ &\quad + p^2 \mathfrak{f}_\chi^2\, \nabla_{p-1}^r \binom{t}{3}\, \varepsilon_{t-2} \frac{B_{t-2,\chi}}{t-2} \,\Big|_{t=l_2'} \pmod{p^r\mathbb{Z}_p}, \end{aligned}$$

where we have already added Euler factors in the last part of the congruence. Applying Lemma 9.8 to the first summand provides that

$$\nabla_{p-1}^r\, \varepsilon_t B_{t,\chi} \,\Big|_{t=l_2'} \equiv -r\nabla_{p-1}^{r-1}\, \varepsilon_t \frac{B_{t,\chi}}{t} \,\Big|_{t=l_2'} \pmod{p^r\mathbb{Z}_p}.$$

With the help of Corollary 5.11 we further obtain that

$$\begin{aligned} -rp^{-(r-1)}\nabla_{p-1}^{r-1}\, \varepsilon_t \frac{B_{t,\chi}}{t} \,\Big|_{t=l_2'} &\equiv r\, \eth^{r-1} L_{p,l}(2,\chi) \\ &\equiv r\, \eth^{r-1} L_{p,l}(0,\chi) \equiv r\, \Delta_{L_{p,l}(\cdot,\chi)}(r-1) \pmod{p\mathbb{Z}_p}. \end{aligned}$$

Now we have to separate the cases. Case $r = 3$: By use of the Kummer congruences the second summand turns into

$$\begin{aligned} \nabla_{p-1}^3 \binom{t}{3} \varepsilon_{t-2} \frac{B_{t-2,\chi}}{t-2} \,\Big|_{t=l_2'} &\equiv -\varepsilon_{l_2'-2} \frac{B_{l_2'-2,\chi}}{l_2'-2} \\ &\equiv \begin{cases} L_{p,p-3}(1,\chi), & (l' < 2), \\ L_{p,l-2}(2,\chi), & (l' \geq 2), \end{cases} \\ &\equiv L_{p,\ell_2}(0,\chi) \pmod{p\mathbb{Z}_p}, \end{aligned}$$

where we have used Lemma 2.6 to derive that

$$\nabla_{p-1}^n \binom{t}{n} = (p-1)^n, \quad t \in \mathbb{Z}_p, n \geq 1. \tag{9.5}$$

Theorem 9.6 and Proposition 9.5 (4) show that

$$L_{p,\ell_2}(0,\chi) \equiv T_{p,\ell_2}(0,\chi) \equiv T_{p,l-2}(0,\chi) \equiv \Delta_{T_{p,l-2}(\cdot,\chi)}(0) \pmod{p\mathbb{Z}_p}.$$

Combining the results from above we finally achieve that

$$\Delta_{T_{p,l}(\cdot,\chi)}(2) \equiv 3\Delta_{L_{p,l}(\cdot,\chi)}(2) + \mathfrak{f}_\chi^2 \Delta_{T_{p,l-2}(\cdot,\chi)}(0) \pmod{p\mathbb{Z}_p}.$$

Case $r = 4$: Using $L_{p,\ell_2}(\cdot,\chi)$ as above, we can write

$$-\nabla_{p-1}^4 \binom{t}{3} \varepsilon_{t-2} \frac{B_{t-2,\chi}}{t-2} \Big|_{t=l_2'} \equiv \nabla_{p-1}^4 \binom{t}{3} L_{p,\ell_2}(s(t),\chi) \Big|_{t=l_2'} \pmod{p^2\mathbb{Z}_p},$$

where we use the variable substitution $s(t) = s_{l_2'} + (t - l_2')/(p-1)$ and $s_{l_2'}$ is a suitable constant depending on l_2'. Since $f = L_{p,\ell_2}(\cdot,\chi) \in \mathcal{K}_{p,2}$, we have

$$f(s) \equiv f(0) + p\,\Delta_f\, s \pmod{p^2\mathbb{Z}_p}, \quad s \in \mathbb{Z}_p.$$

Thus we obtain

$$\begin{aligned}
\nabla_{p-1}^4 \binom{t}{3} f(s(t)) \Big|_{t=l_2'} &\equiv \nabla_{p-1}^4 \binom{t}{3} \left(f(0) + p\,\Delta_f \left(s_{l_2'} + \frac{t-l_2'}{p-1} \right) \right) \Big|_{t=l_2'} \\
&\equiv \left(f(0) + p\,\Delta_f \left(s_{l_2'} + \frac{3-l_2'}{p-1} \right) \right) \nabla_{p-1}^4 \binom{t}{3} \Big|_{t=l_2'} \\
&\quad + p\,\frac{4\Delta_f}{p-1} \nabla_{p-1}^4 \binom{t}{3} \frac{t-3}{4} \Big|_{t=l_2'} \\
&\equiv -p\,4\Delta_f \pmod{p^2\mathbb{Z}_p},
\end{aligned}$$

where the first summand vanishes by Lemma 2.6 and the second summand is reduced by (9.5) observing that $\binom{t}{4} = \binom{t}{3}\frac{t-3}{4}$. By Theorem 9.6 and Proposition 9.5 (5) we have

$$2\Delta_f \equiv 2\Delta_{L_{p,\ell_2}(\cdot,\chi)} \equiv \Delta_{T_{p,\ell_2}(\cdot,\chi)} \equiv \Delta_{T_{p,l-2}(\cdot,\chi)} \pmod{p\mathbb{Z}_p}.$$

Putting all together, we finally get

$$\Delta_{T_{p,l}(\cdot,\chi)}(3) \equiv 4\Delta_{L_{p,l}(\cdot,\chi)}(3) + 2\mathfrak{f}_\chi^2 \Delta_{T_{p,l-2}(\cdot,\chi)} \pmod{p\mathbb{Z}_p}. \qquad \square$$

As a result, we get criteria for exceptional primes regarding $L_{p,0}(\cdot,\chi)$ and for the existence of unique simple zeros of the corresponding functions $\tilde{L}_{p,0}(\cdot,\chi)$ in case χ is odd.

Corollary 9.12. *Let $\chi \in \mathfrak{X}_2$ and $p > 3$ where χ is odd and $\chi(p) = 1$. For $f = L_{p,0}(\cdot,\chi)$ we have the following statements:*

(1) *We have*

$$T_{p,0}^2(0,\chi) \equiv 0 \pmod{p\mathbb{Z}_p} \quad \Longleftrightarrow \quad (p,0) \in \Psi_\chi^{\mathrm{exc}}.$$

(2) *If $(p,0) \in \Psi_\chi^{\mathrm{exc}}$, then*

$$T_{p,0}^3(0,\chi) \not\equiv \mathfrak{f}_\chi^2\, T_{p,p-3}(0,\chi) \pmod{p\mathbb{Z}_p} \quad \Longleftrightarrow \quad \lambda_f = 2.$$

If $\lambda_f = 2$, then $\tilde{L}_{p,0}(\cdot,\chi)$ has a unique simple zero.

Proof. This is a consequence of Proposition 9.5 and Theorems 7.19, 9.6, and 9.11. $\square$

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